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“Advanced Statistics and Data Analysis”

Syllabus

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Intelligent Transport Systems: New ICT – based Master’s Curricula in Uzbekistan (INTRAS)
Agreement number: 2017-3516/001-001
Project reference number: 586292-EPP-1-2017-1-PL-EPPKA2-CBHE-JP

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GENERAL OVERVIEW OF THE LECTURE:

- Understanding what is a deterministic system
- Understanding what is a stochastic system
- Understanding the importance of the probability analysis
- Learn how to construct a probability distribution for a random variable
- Learn how to calculate the average, the mode, the mean, the variance, and expected value for a discrete random variable.
- Learn how to plot the probability- and the cumulative- distribution functions of stochastic variables/scenarios
- Learn how to identify the properties of the normal distribution.
- Learn how to find probabilities for a normally distributed variable by transforming it into a standard normal variable.
- Learn how to find specific data values for given percentages, using the standard normal distribution: The Resident- and Traveling- Times Calculation.
- Learn how to apply the "Central Limit Theorem" to solve problems involving "sample means" for large samples
- Understanding the concept of „QUEUING"
- Understanding the interest of „QUEUING" in the processing of stochastic scenarios.
- Learn how to use a software/toolbox for the analysis of a „Queuing Process".

Chapter 1.

General Introduction

(Chapter's detailed description)

1.1. Importance of statistics and data analysis in road and railway transportation as well as in supply chain networks

- ✓ Explaining why statistics is important in the field of transportation (i.e. Railway transportation, Road transportation, and Supply chain networks).
- ✓ Explaining why data analysis is important in the field of transportation.
- ✓ Give sample examples to illustrate the two concepts defined.

1.2. Definition of some important keywords and their illustration through concrete examples

- ✓ Definition (with concrete illustrative examples) of the following keywords and concepts: Statistics, Advanced statistics, Data, Data analysis, deterministic, stochastic, distribution functions, mean/average, and data forecasting

1.3. Commonly used methods, concepts and algorithms for data analysis

- ✓ Explaining why statistics is important in the field of transportation (i.e. Railway transportation, Road transportation, and Supply chain networks).
- ✓ Explaining why data analysis is important in the field of transportation.
- ✓ Give sample examples to illustrate the two concepts defined.

1.4. Commonly used methods, concepts and algorithms for data forecasting

- ✓ Presenting and describing the commonly, concepts and algorithms for data forecasting. Give specific application examples for illustration.

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Chapter 2.

Statistical analysis of stochastic phenomena

(Full chapter presentation)

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2.0.0. Contents

- **Introduction**
  - What is a deterministic system?
  - What is a stochastic system?
  - Deterministic formalism Vs. Stochastic formalism
  - Why is the estimation process important?
  - What is the essence of an estimation process?

- **Fundamental parameters of stochastic processes/scenarios**
  - Mean
  - Median
  - Mode
  - Variance
  - Standard deviation
  - Covariance
  - Percentile
  - Quartile
  - Confidence interval (CI)

2.0.0. Contents

- **Application example**
- **Estimation of the Confidence interval (CI)**
- **Elements of the CI**
- **Confidence limit for population mean**
- **The central limit theorem**
- **Normal distribution Vs. Standard normal distribution**
- **Level of confidence**
- **Interval and level of confidence**
- **Factors affecting the CI range**
- **Confidence interval estimates**
- **Z-score Vs. Measurement scale**
- **How to exploit the CI**
- **Application example**
- **The normal distribution**
- **Concluding remarks**

2.0.1 Objectives

- **Understanding what is a deterministic system**
- **Understanding what is a stochastic system**
- **Understanding the importance of the probability analysis**
- **Learn how to construct a probability distribution for a random variable**
- **Learn how to calculate the average, the mode, the mean, the variance, and expected value for a discrete random variable.**
- **Learn how to plot the probability- and the cumulative- distribution functions of stochastic variables/scenarios**
- **Learn how to identify the properties of the normal distribution.**

2.0.1 Objectives

- **Learn how to find probabilities for a normally distributed variable by transforming it into a standard normal variable.**
- **Learn how to find specific data values for given percentages, using the standard normal distribution: The Resident- and Traveling- Times Calculation.**
- **Learn how to apply the "Central Limit Theorem" to solve problems involving "sample means" for large samples**
- **Understanding the concept of „QUEUING"**
- **Understanding the interest of „QUEUING" in the processing of stochastic scenarios.**
- **Learn how to use a software/toolbox for the analysis of a „Queuing Process".**

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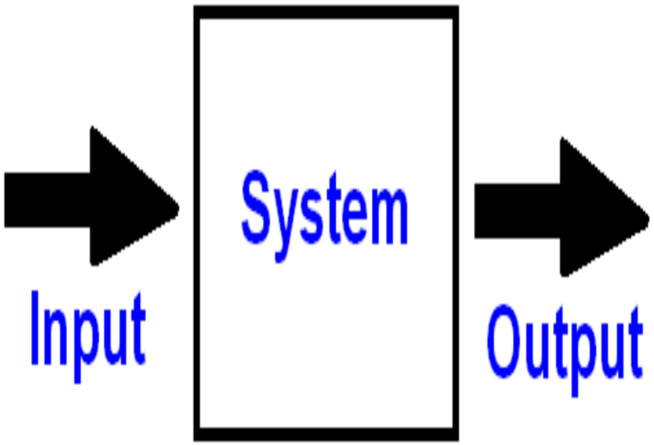
2.1. Deterministic systems, stochastic systems and some illustrative examples in transportation (1)

What is a deterministic system?

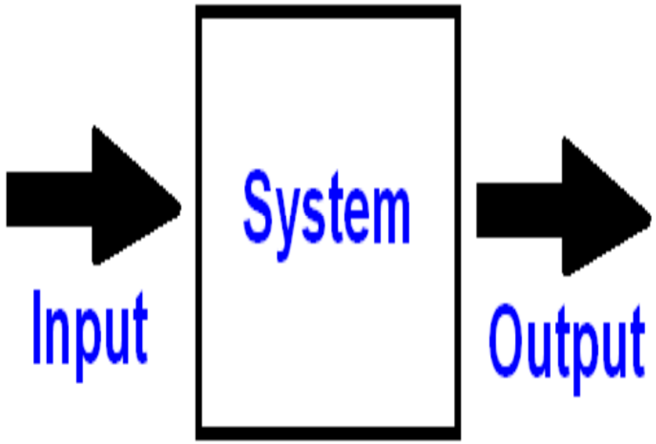
- A system that produces the same output for a given starting condition (or input).
- A system in which the next state is uniquely determined by the current state.
- A system with a fully predictable state/Behavior (i.e. no randomness is observed in the behavior of the system).
  - ✓ **Examples:**
    - Classical Mechanics: Pendulum, motors, etc…
    - Electrical engineering: Electrical- and Electronic circuits, etc..



2.1. Deterministic systems, stochastic systems and some illustrative examples in transportation (2)

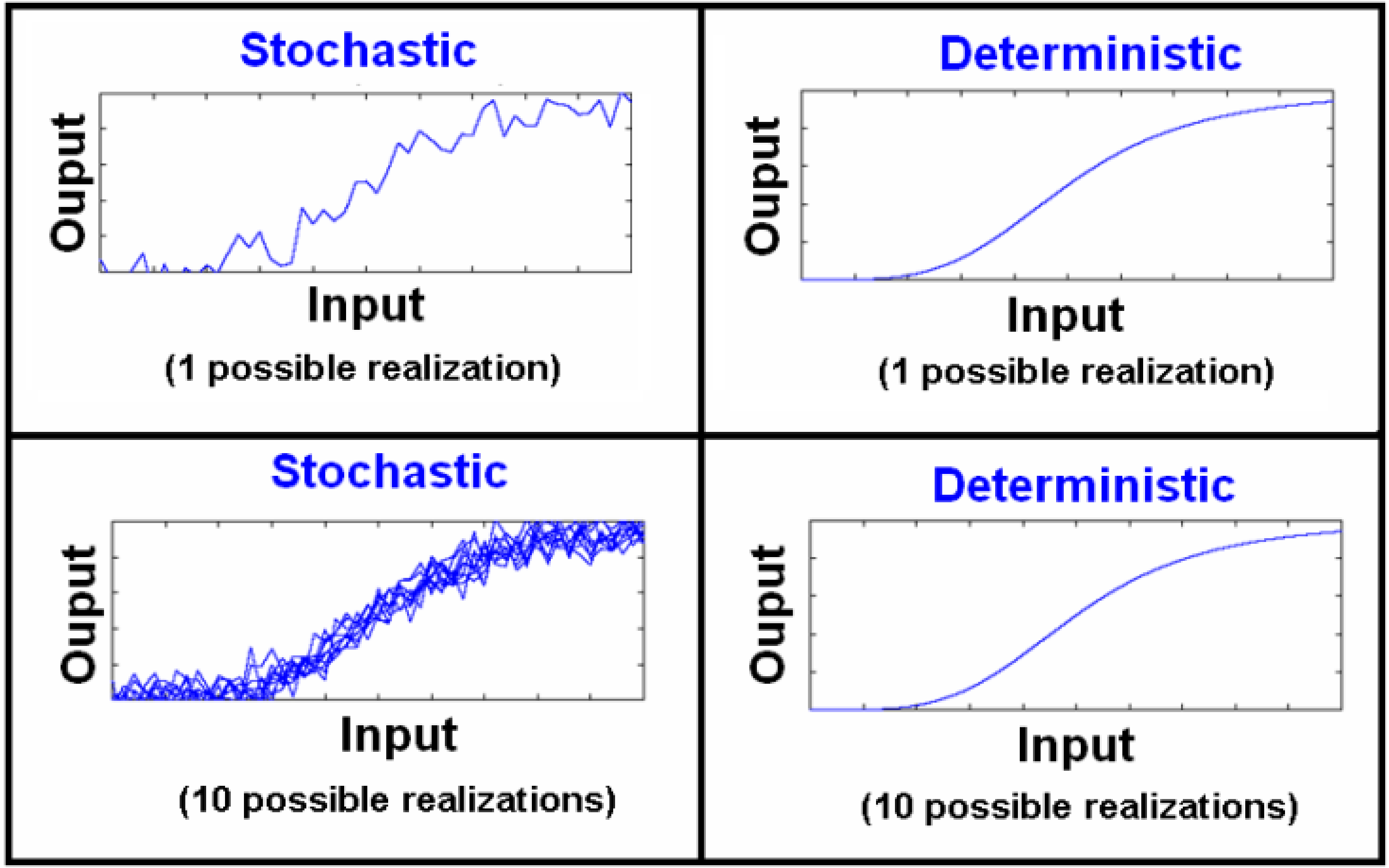
What is a stochastic system?

- A system that produces different output for a given starting condition (or input) (Note: this is observed through repeated measurements).
- A system in which the next state cannot be predicted/determined by the current state.
- A system in which the next state (output) is only probabilistically determined by the current state (i.e. there are several possible next states that can occur subsequent to the same activity (input), each with a given probability.



✓ **Examples of stochastic phenomena:**
  - Phenomena/scenarios in transportation and traffic Engineering.
  - Phenomena in Quantum Mechanics, etc..

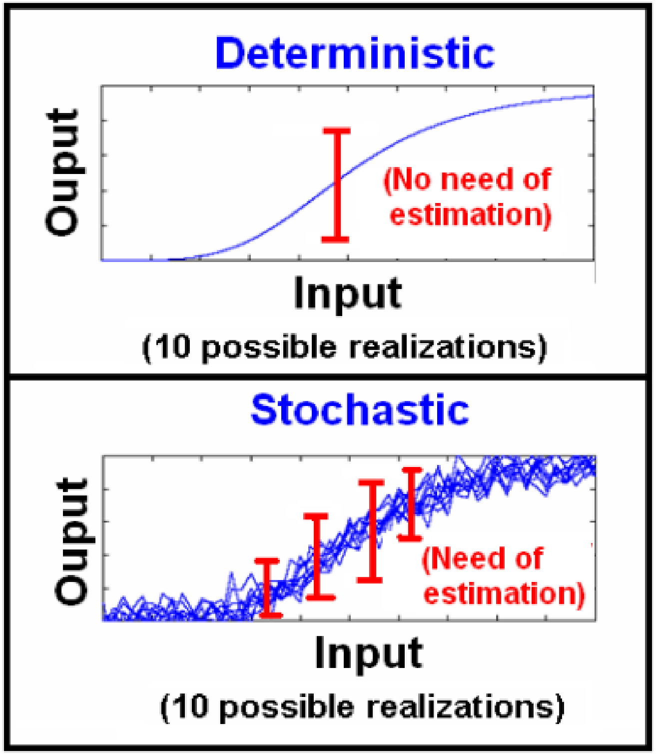
2.2. Deterministic formalism Vs. Stochastic formalism



2.3. Importance and essence of the estimation process in data analysis (1)

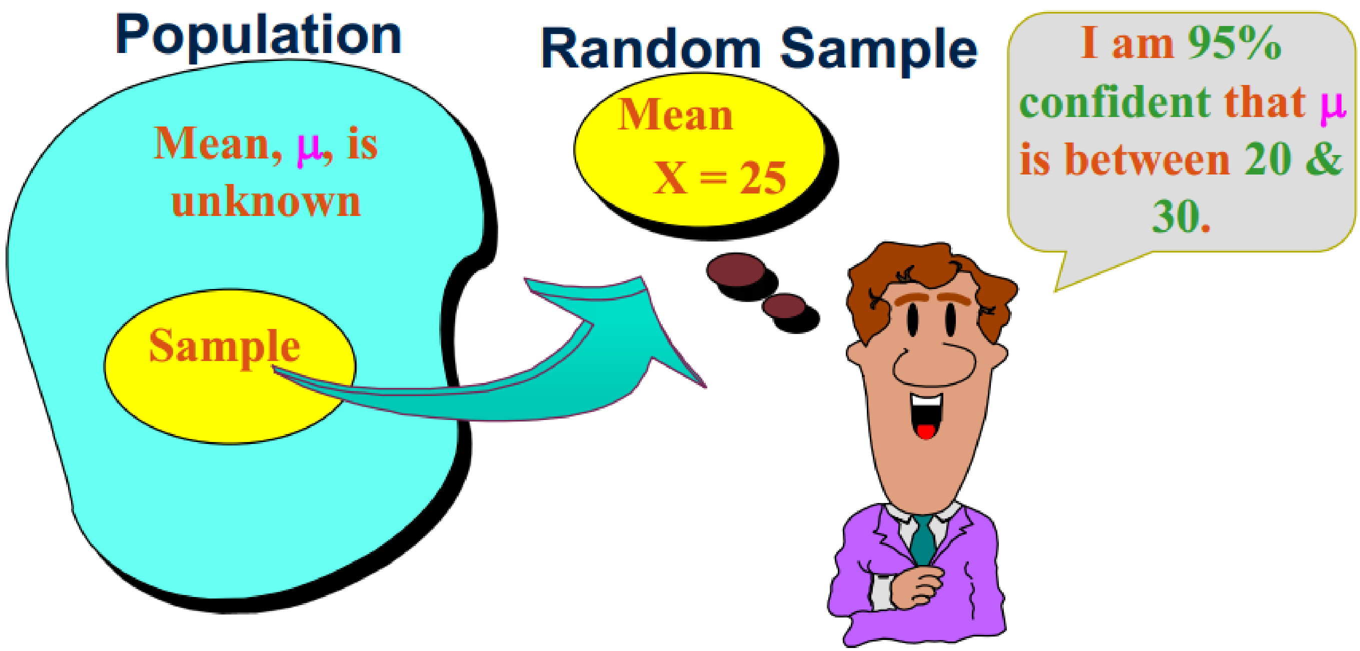
Why is the estimation process important ?

- To analyze stochastic (or random) scenarios/events.
- The result of the „estimation process“ provides a range around the „Mean“. This range generally reveals the domain in which the expected value can be located. This range, which is called „Confidence Interval“ (CI) is obtained at a specific level of confidence.



2.3. Importance and essence of the estimation process in data analysis (2)

What is the essence of an „Estimation Process” ?

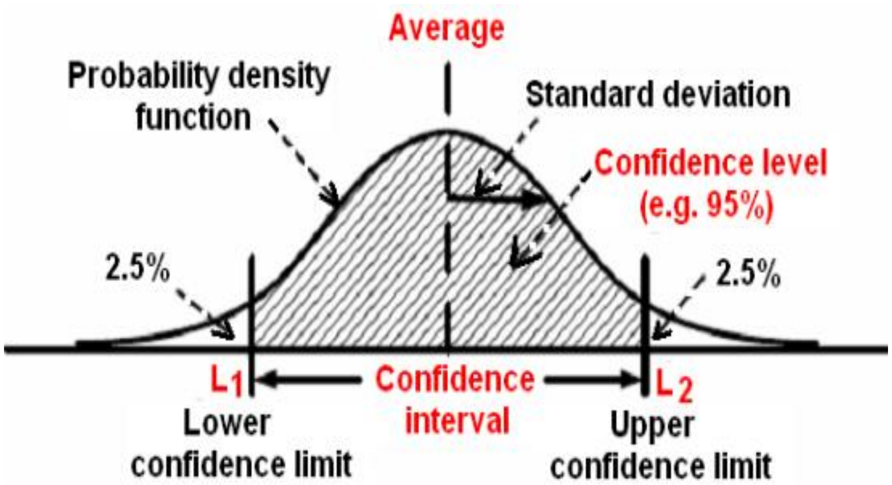


2.4. Fundamental parameters of a stochastic process and measurement (1)

What are the fundamental/key parameters for a "Population Estimation"?

- The fundamental parameters are key quantities in the analysis, prediction and control of stochastic processes/scenarios

- Cases of Univariate/Multivariate processes
  - ✓ Mean
  - ✓ Median
  - ✓ Mode
  - ✓ Variance /Covariance
  - ✓ Standard deviation
  - ✓ Quantile
  - ✓ Confidence interval



2.4. Fundamental parameters of a stochastic process and measurement (2)

What are the fundamental/key parameters for a “Population Estimation”?

- ❑ Mean: $$\mu = \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$
- ❑ Mode/Top: $$x_{\mathrm{mod}} = \arg\max_{x} \hat{f}(x)$$
- ❑ Median $$x_{med} = \begin{cases} x_{((n+1)/2)} & \text{if } n \text{ is odd} \\ \dfrac{x_{(n/2))+} x_{((n/2)+1))}}{2} & \text{Otherwise} \end{cases}$$
- ❑ Variance: $$Var(X) = E\left[(X-\mu)^2\right]$$

2.4. Fundamental parameters of a stochastic process and measurement (3)

What are the fundamental/key parameters for a "Population Estimation"?

- Standard deviation

$$\sigma = \sqrt{E\left[(X-\mu)^2\right]} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2}$$

- Confidence interval

$$CI = \mu \pm Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

- Covariance

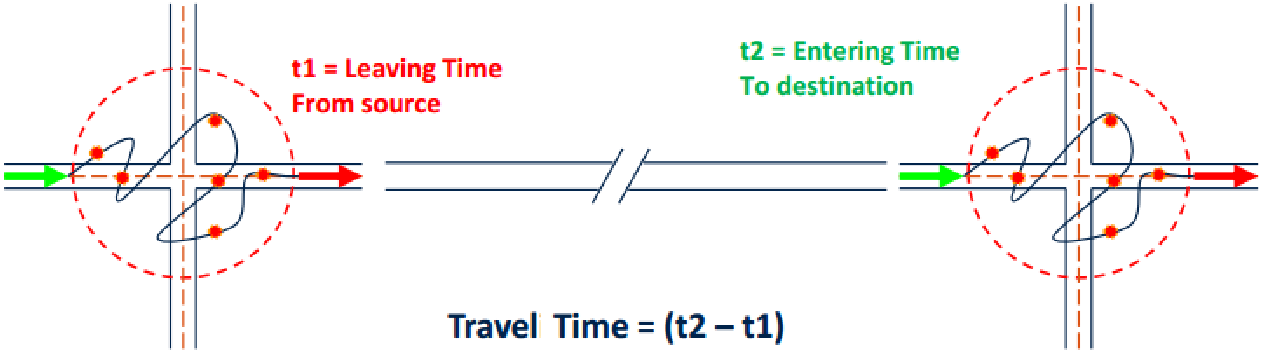
$$Cov(X,Y) = E(XY) - \mu_X . \mu_Y$$

$$\mu_X = E(X) = \int_{-\infty}^{+\infty} xf(x)dx$$

$$E(X) = \sum_{i=1}^{n} x_i . p_i$$

2.4. Fundamental parameters of a stochastic process and measurement (4)

- Exercise
  - ✓ The table below gives the travel time of vehicles between two junctions/ nodes (or the travel time of trains between two neighboring stations).
    - Calculate the Mean, the variance, the standard deviation and the mode of the distribution.

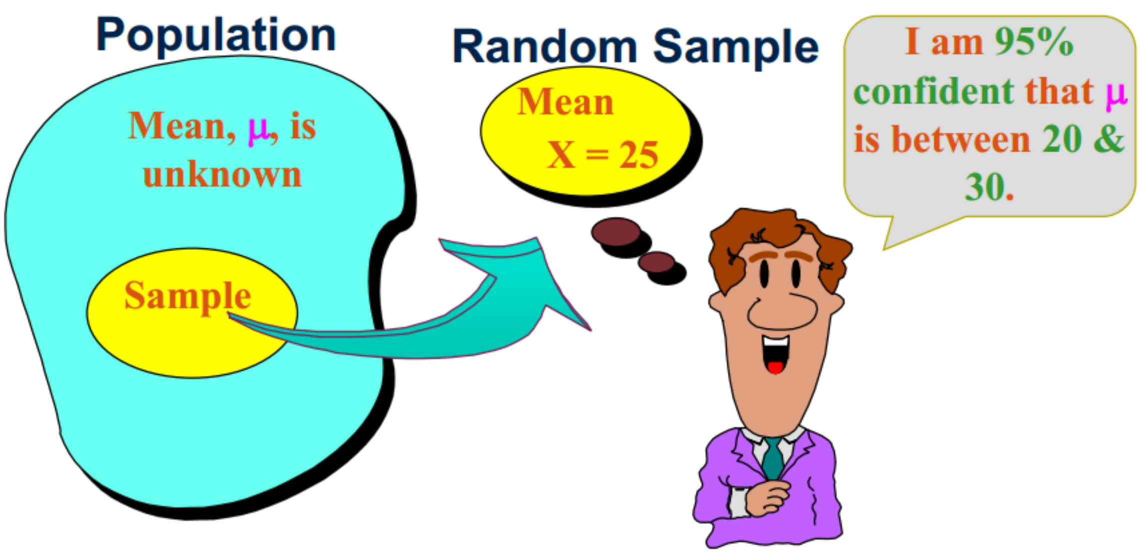


Note: This table contains fictitious data

Number of cars/trains	1	5	6	8	6	3	1
Travel time (s)	10	20	25	30	40	50	80

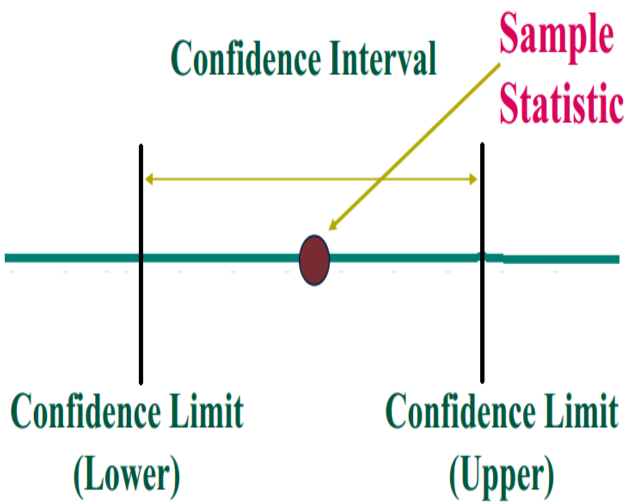
2.5. Estimation of the confidence interval (CI)

- **The estimation of the confidence interval -**
  - ✓ Provides range of values based on observations from 1 sample . This range is stated in terms of probability (because never 100% sure)
  - ✓ Gives information about closeness to unknown population parameter
  - ✓ A probability that the population parameter falls somewhere within the interval.



2.6. Elements of the "confidence interval estimation" and importance of the Z-score value

- **The elements are:**
  - ✓ The confidence level
  - ✓ The mean
  - ✓ The standard deviation
  - ✓ The sample size
  - ✓ The confidence limits
  - ✓ The Z-score values
- **Importance of Z-score**
  - ✓ The value of Z-score significantly affects the length of the confidence interval
  - ✓ The Z-score value is obtained for a fixed/known value of the quantile. Various tables of Z-score are available in the literature and each value of a quantile is used to obtain the corresponding Z-score value.



2.7. Confidence limits for population mean and importance of the confidence interval

- **Confidence limits:**

$$\text{Parameter} = \text{Statistic} \pm \text{Its } \mathit{Error}$$

$$\mu = \overline{X} \pm \mathit{Error}$$

$$\overline{X} - \mu = \mathit{Error} = \mu - \overline{X}$$

$$Z = \frac{\overline{X} - \mu}{\sigma_{\overline{X}}} = \frac{\mathit{Error}}{\sigma_{\overline{X}}}$$

$$\mathit{Error} = Z\sigma_{\overline{X}}$$

$$\mu = \overline{X} \pm Z\sigma_{\overline{X}}$$

- **Importance of the confidence interval**
  - ✓ The confidence interval provides the range/window in which the expectation is located. This range contains the acceptable values of the random variable.

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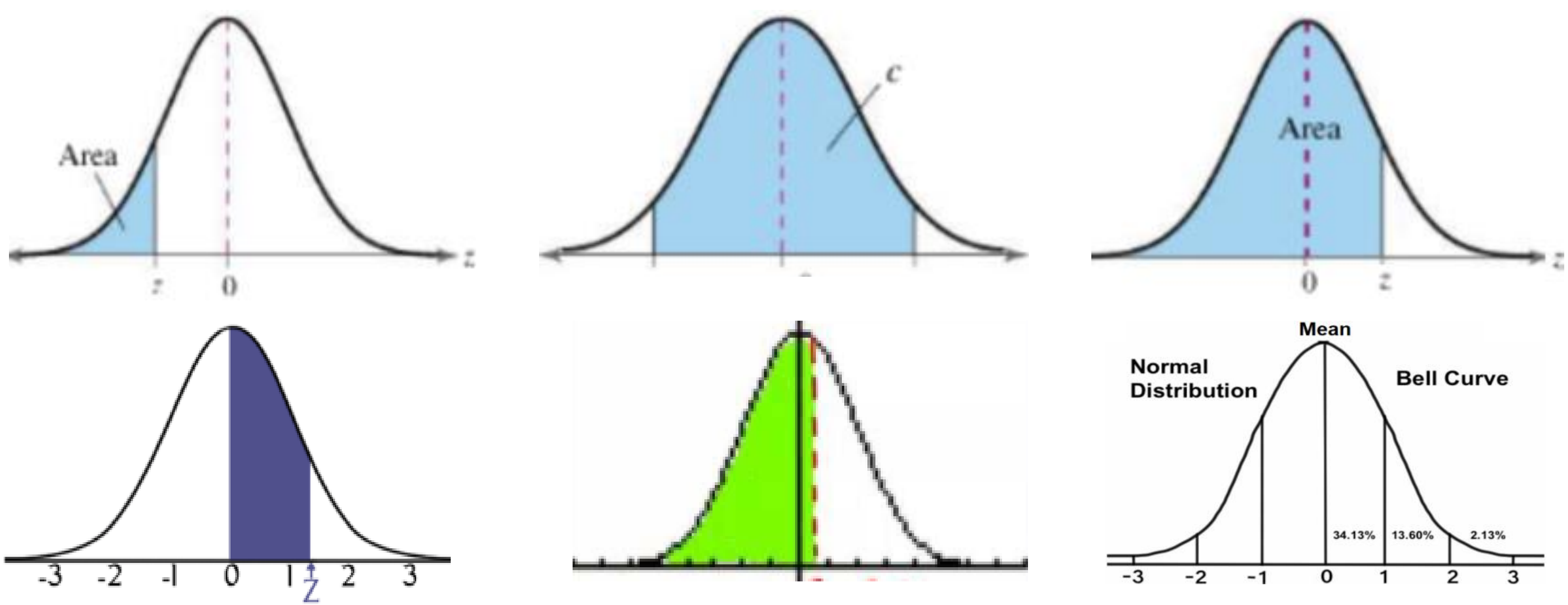
2.8. Z-score tables: Description, reading techniques, and importance in data analysis

- ❑ Learning how to adapt the value of a quantile to a specific table of Z-score.
- ❑ Learning how to read the various Z-score tables available in the literature
- ❑ Each area of the figures below corresponds to a specific table of Z-score
- ❑ For a given/known value of a quantile the corresponding value of Z-score is the same when reading through all tables defined by the figures below. This statement must be clearly explained through concrete application examples.



2.9. The central limit theorem (CLT)

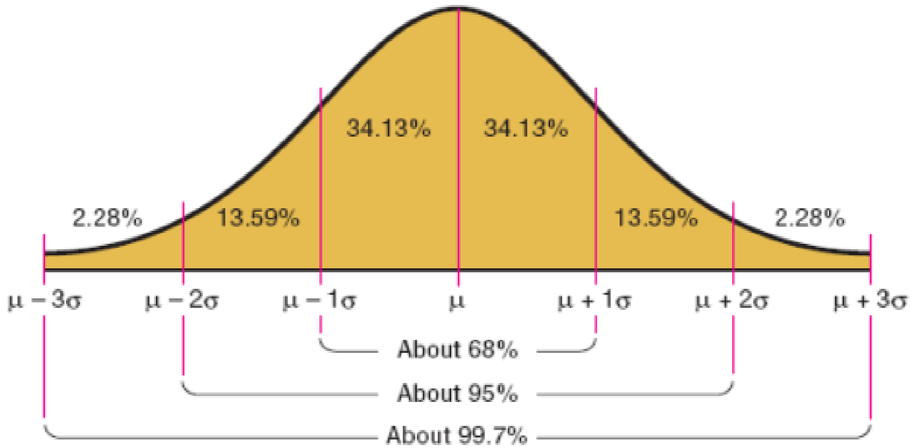
- Let $(X_1, \ldots\ldots\ldots\ldots, X_n)$ be a simple random sample from a population with mean μ and variance σ^2.
- Let $\overline{X} = (X_1, \ldots\ldots\ldots\ldots, X_n)/n$ be the sample mean.
- Let $S_n = (X_1, \ldots\ldots\ldots\ldots, X_n)$ be the sum of the sample observations.
- Then if „n" is sufficiently large,

  $$\overline{X} \cong N\left(\mu, \frac{\sigma^2}{n}\right) \text{ and } S_n \cong N(n\mu, \sigma^2)$$ approximaely
- For most populations, if the sample size is greater than "30", the central limit theorem approximation is good.
- Thus, the appropriate formula for the Z value is $Z = \dfrac{\overline{X} - \mu}{\sigma / \sqrt{n}}$

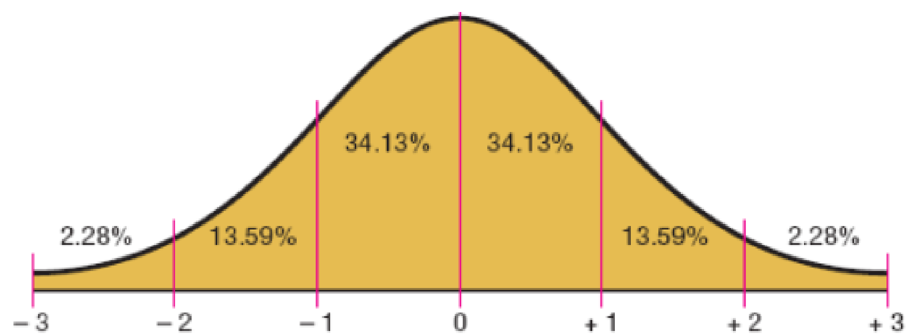
2.10. Normal distribution versus standard normal distribution (Z-score)

- The **standard normal distribution** is a normal distribution with a mean of 0 and a standard deviation of 1

Area Under the Normal Distribution Curve



Area Under the Standard Normal Distribution Curve



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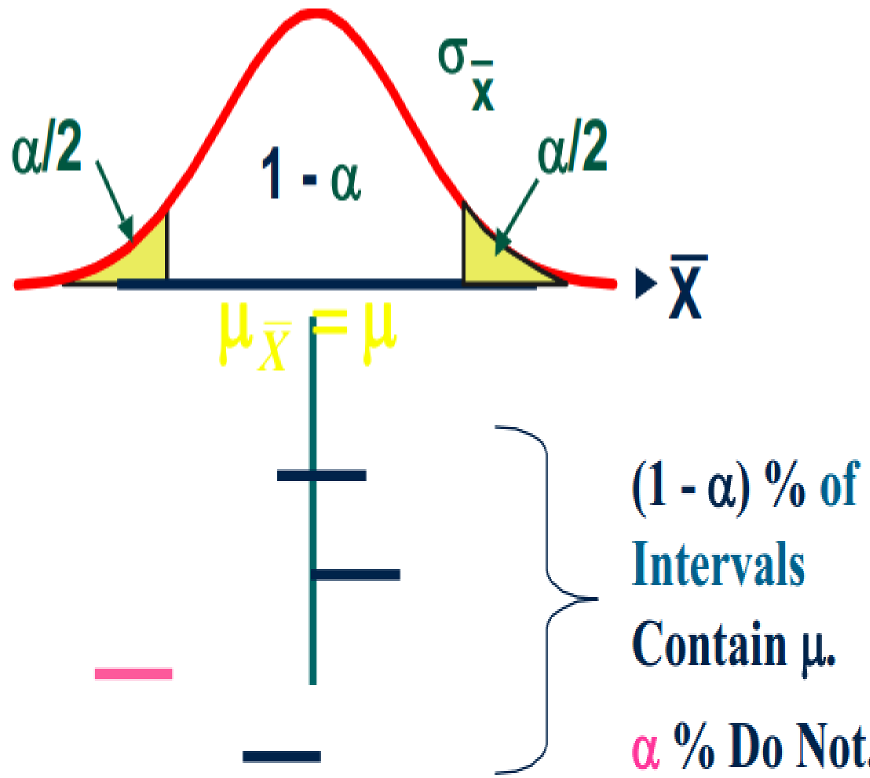
2.11. Factors affecting the confidence interval (CI) range (1)

- **Level of confidence**
  - ✓ Probability that the unknown population parameter is in the confidence interval in 100 trials.
  - ✓ Denoted (1-α)% = level of confidence (e.g., 90%, 95%, 99%, etc.). The quantity α is the probability that the unknown population parameter is not within the interval in any other 100 of trials performed.



$$CI = \left[\bar{X} - Z\frac{\sigma}{\sqrt{n}} \quad \bar{X} + Z\frac{\sigma}{\sqrt{n}}\right]$$

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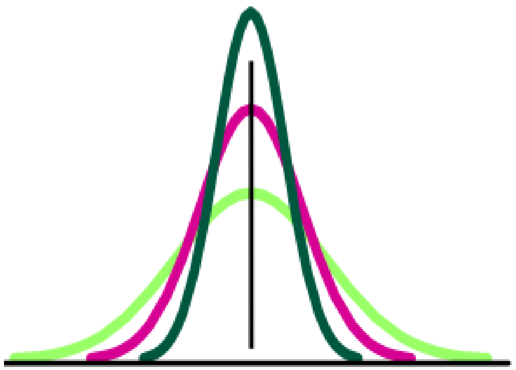
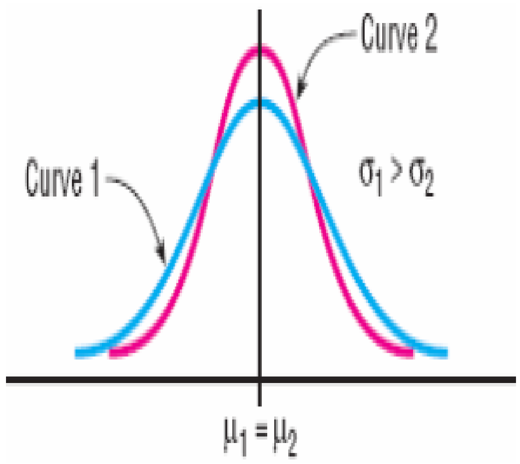
2.11. Factors affecting the confidence interval (CI) range (2)

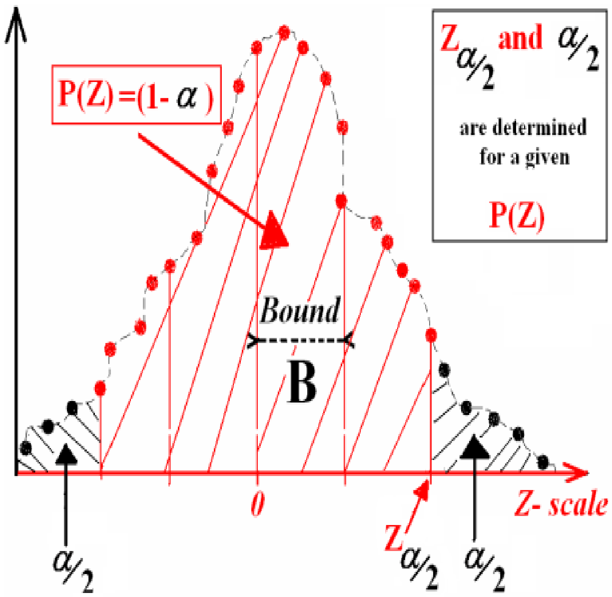
- **Data variation** σ_X
- **Sample size** $\sigma_{\overline{X}} = \sigma_X / \sqrt{n}$
- **Level of confidence** $(1-\alpha)$
- **Confidence interval** $CI = \left[\overline{X} - Z\frac{\sigma}{\sqrt{n}} \quad \overline{X} + Z\frac{\sigma}{\sqrt{n}}\right]$
- **Figures are with the same mean but different standard deviations**





2.12. Confidence interval "estimates" under some assumptions

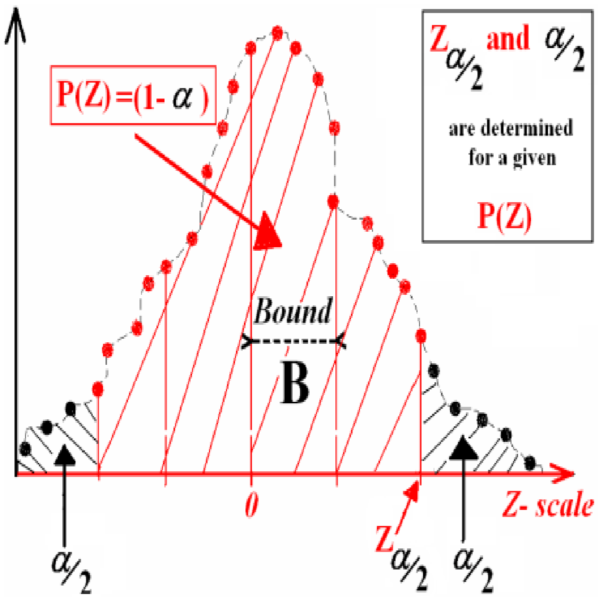
- Assume the population standard deviation is known
- Assume the population is normally distributed. If not, we must use a large sample data.
- Under these assumptions, the CI is estimated as follows:



$$\text{CI} = \mu - \left(\frac{\sigma}{\sqrt{n}}\right) * Z_{\left(\frac{\alpha}{2}\right)} \le \mu_{\text{expected}} \le \mu + \left(\frac{\sigma}{\sqrt{n}}\right) * Z_{\frac{\alpha}{2}}$$

2.13. Z-score versus measurement scale

- **What is a Quantile ?**
  - ✓ A percentage expected for a given distribution.
    - Example of quantiles:
      - 100*P(Z)=10%;
      - 100*P(Z)=50% (Median);
      - 100*P(Z)=68%;
      - 100*P(Z)=95%;



- **How to determine $Z_{\alpha/2}$?**
  - ✓ **Step 1:** Definition of the precision we want to achieve when analyzing data which average is denoted μ.
  - ✓ **Step 2:** Represent P(Z) in the plan Z-score.
  - ✓ **Step 3:** Use the value of P(Z) in the well-known table of Z-score to determine $Z_{\alpha/2}$

2.14. How to exploit confidence level- values in the table of z-score ?

Quantiles: 100*P(Z)	Z (‹/ 2)
10%	0.13
30%	0.39
50%	0.675
68%	0.99
90%	1.645
95%	1.96
99%	2.575

$$\text{CI} = \mu - \left(\frac{\sigma}{\sqrt{n}}\right) * Z_{\left(\frac{\alpha}{2}\right)} \leq \mu_{\text{expected}} \leq \mu + \left(\frac{\sigma}{\sqrt{n}}\right) * Z_{\frac{\alpha}{2}}$$

2.15. Application example

- Consider the exercise above (see section 2.4) and determine the confidence interval on the travel time between the two junctions/nodes (or the travel time of trains between two neighboring stations) for each of the following level of confidence:
    - 10%;
    - 40
    - 50% (Median);
    - 60
    - 68%;
    - 80
    - 95%;
- Comment the results obtained

2.16. The normal distribution (1)

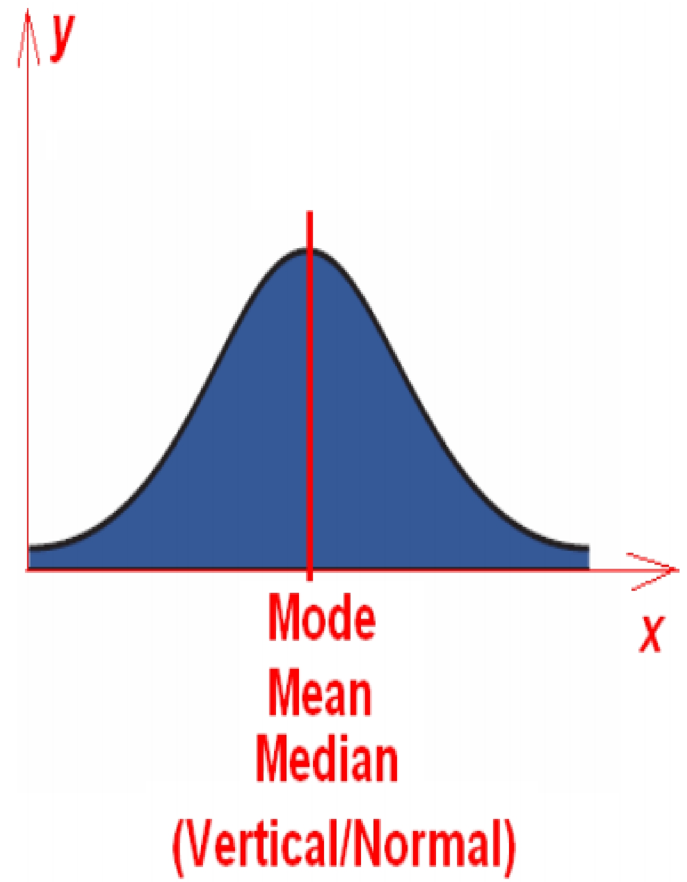
- What is the normal distribution?
  - ✓ The Gaussian distribution [Carl Friedrich Gauss (1777–1855)] defined as follows (where X is a continuous variable):

$$X \cong N\left(\mu,\ \sigma^2\right)$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < +\infty$$

2.16. The normal distribution (2)

- **Normal Probability of the** Gaussian distribution
  - ✓ The mean, median, and mode are equal and located at the center of the distribution.
  - ✓ The curve is unimodal (i.e., it has only one mode).
  - ✓ The curve is symmetric about the mean, (its shape is the same on both sides of a vertical line passing through the center.
  - ✓ The curve is continuous, (there are no gaps or holes). For each value of *X*, there is a corresponding value of Y.
  - ✓ The total area under the normal distribution curve is equal to 1.00, or 100%.



2.17. Concluding remarks

- Statistical Inference involves drawing a sample from a population and analyzing the sample data to learn about the population.
- In many situations, one has an approximate knowledge of the probability mass function or probability density function of the population.
- In these cases, the probability mass or density function can often be well approximated by one of several standard families of curves or function discussed in this chapter.
- The confidence interval is strongly correlated to/with both the form of the distribution function (Gaussian, Poisson, Markov, etc...) exhibited by a stochastic scenario and the sample size of the stochastic scenario. These multiple forms of the distribution have a significant impact on the queuing policy (at server level) to manage the stochastic processes.

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Chapter 3.

Basics of traffic theory: Fundamentals of queuing and simulation of queuing processes in stochastic scenarios/Events

(Full chapter presentation)

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3.0. Contents

- **Overview of traffic processes/phenomena**
  - ✓ What is a traffic?
  - ✓ What are the main traffic problems?
  - ✓ What is traffic theory?
  - ✓ What is the aim of traffic theory?
  - ✓ What is a stochastic phenomenon/process?
  - ✓ What is a random variable?
  - ✓ Examples of random variables
  - ✓ What is a Poisson process?
  - ✓ What is a Poisson distribution?
  - ✓ Why the Poisson distribution?
  - ✓ Why the mathematical probability distribution functions?
  - ✓ What are the main limitations of the stochastic theory?
  - ✓ What are the methods used in traffic theory?
  - ✓ What are the limits of these methods?

3.0. Contents

- **Probability distributions**
  - ✓ Exponential distribution (Markov process)
  - ✓ Shifted exponential distribution
  - ✓ Poisson distribution
  - ✓ Relation between Poisson and exponential distribution
  - ✓ Erlang distribution

- **Motivation and overview of queuing**
  - ✓ What is a queuing system?
  - ✓ Why queuing theory?
  - ✓ Some applications of queuing
  - ✓ Simple examples of the use of queuing theory in networking
  - ✓ Queuing theory answers questions like
  - ✓ First illustrative example where queuing theory is being used
  - ✓ Second illustrative example where queuing theory is being used
  - ✓ Fundamentals of Queuing Systems

3.0. Contents

- **General queuing notation**
  - ✓ Five components of a Queuing system
  - ✓ These five components influence the general queuing notation (Kendall) A/B/m/N –S
  - ✓ Kendall notation: X/Y/m/k
  - ✓ The interarrival time random variable
- **Some queuing models/systems**
  - ✓ M/M/1 Queue
  - ✓ D/D/1 Queue
  - ✓ M/D/1 Queue
  - ✓ G/G/1 Queue

3.0. Contents

- **State analysis of queue models/systems**
  - Poisson's law in Physics
  - Poisson's law in operational research
  - Poisson's law in communications
  - Poisson's law in transportation
  - Single-server queuing system
  - State
  - Probability of given state
  - Similar to AC
  - Derivation
  - Probability sum equal 1
  - Geometric sum
  - Mean value
  - Differentiation wrt specific variables
  - Application: Case of M/M/1 queue
  - Homework: Proposal of concrete applications

3.0. Contents

- **Blocking queuing models/systems**
  - ✓ What is an Erlang?
  - ✓ Exercises - Solutions - Remarks
  - ✓ What is the goal of Erlang traffic measurements?
  - ✓ What are the advantages of Erlang traffic models?
  - ✓ What are the types of Erlang traffic models?
  - ✓ What are the uses of the Erlang B traffic model?
  - ✓ What are the uses of the extended Erlang B traffic model?
  - ✓ What are the uses of the Erlang C traffic model?
  - ✓ The Erlang B distribution
  - ✓ The Erlang C distribution
  - ✓ What is the Busy Hour Traffic (BHT)?
  - ✓ What is blocking describes
  - ✓ What are the key factors in line estimation?
  - ✓ Brief description of VoIP and Queuing models in VoIP
  - ✓ Design process of communications networks
  - ✓ Proposal of some concrete examples

3.1. Definition of keywords (1)

- **What is Traffic?**
  - ✓ A transportation of:
    - Messages/information (telecommunications)
    - Persons (Road traffic & Railway traffic)
    - Goods/information (Supply chain networks)
    - Etc…
- **What is traffic theory?**
  - ✓ Analysis of traffic phenomena/events/issues
- **What is a stochastic phenomenon/process?**
  - ✓ A random phenomenon
  - ✓ A sequence of random variables
- **What is a random variable?**
  - ✓ A function defined on a sample space and about whose values a probability statement can be made.
- **What is a Poisson process?**
  - ✓ A stochastic process which is defined in terms of the occurrences of events
  - ✓ A counting process which is given as a function of time N(t), to represent the number of events since time t = 0

3.1. Definition of keywords (2)

- **What is a Poisson distribution?**
  - ✓ A discrete probability distribution
    - E. g.: The number of events between time a and time b is given as N(b) - N(a) and has a Poisson distribution
- **Why the Poisson distribution?**
  - ✓ To express the probability of number of events occurring in a fixed period if these events
    - occur with a known average rate, and
    - are independent of the time since the last event.
- **Why the mathematical probability distribution functions?**
  - ✓ To describe analytically stochastic processes in traffic scenarios.
- **What are the main limitations of the stochastic theory?**
  - ✓ The stochastic theory is very difficult due to the following aspects:
    - The large number of stochastic variables
      - case of road and rail traffic: velocity, dynamic behavior, route choice, etc
  - ✓ The stochastic theory is very difficult due to the following aspects:
    - The large dependencies between the stochastic variables
      - case of railway and road traffic: The use of a simple distribution function is not possible; a complex one is required

3.2. Methods used in traffic theory: Pros/Advantages and Cons/drawbacks

- Deterministic methods
- Stochastic (non-deterministic) methods
- In both cases the most common approaches are:
  - ✓ Analytical
  - ✓ Graphical (e.g. the determination of the railway capacity)
  - ✓ Simulation
- **What are the limits of these methods? (1)**
  - ✓ Some traffic systems/processes are too complex to be modeled in mathematical form.
    - A flexible control of a complicated road/rail intersection
    - The influence of delays, caused by failures, on the time- table of a railway system
    - The dynamic behavior of traffic flows
  - ✓ Some traffic systems/processes can be modeled in mathematical form
    - Exact analytical solutions are impossible
    - Approximated analytical solutions are possible but do not provide accurate information on the systems/processes/events
      - Example: Complicated queuing systems

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3.2. Definition of keywords

- **What are the limits of these methods? (2)**
  - ✓ Studying of the real process can be impossible or too expensive
    - E.g.:
      - Flight simulators for the training of pilots
      - Studying the effects of road pricing
- **What are the advantages of these methods?**
  - ✓ Simulation method can tackle/overcome some of the limitations underlined above. Some important features of simulation are:
    - The possibility of studying a system with complicated internal reactions
    - The sensitivity of the traffic flow parameters/variables can be analyzed easily.
    - Parameters/variables that are important and/or significantly affect traffic flows can be found
    - The interaction between these parameters/variables can be easily controlled
    - The dynamical behavior of traffic flows can be studied with a different time scale: accelerated or in slow motion (case of road traffic)

3.3. Models used in stochastic theory: Pros/Advantages and Cons/drawbacks

- **Models used in stochastic theory are expressed in terms of distribution functions**
  - ✓ Exponential
  - ✓ Shifted exponential
  - ✓ Poisson
  - ✓ Erlang
  - ✓ General
- **Advantages of stochastic models**
  - ✓ They allow the assignment of probabilities to any event you want to. Specifically, we do not limit ourselves to those events that are repeatable.
  - ✓ Probability distribution functions (PDF) define the output of the function which tells us the likelihood of certain outcomes in a random process.
  - ✓ All PDF share a lot of similarities with true randomness.
- **Drawbacks of stochastic models**
  - ✓ The main drawback is that the models are not purely objective or reliable. This is due to the degree of randomness which cannot be accurately quantified for given random scenarios, processes, events, etc.
  - ✓ Mathematical functions (i.e., PDF) are approximation of the unknown reality.

3.4. Some mathematical probability distribution functions (1)

- Mathematical probability functions are often used to describe stochastic processes in traffic scenarios. Below are some commonly used models.
- **Exponential distribution (Markov process)**
  - ✓ The Cumulative distribution function (cdf) F(x;λ): this is the basis for a probability distribution.

$$F(x;\lambda) = \begin{cases} 1 - e^{-\lambda x} & , x \geq 0, \\ 0 & , x < 0. \end{cases} \qquad \lim_{x \to -\infty} F(x;\lambda) = 0 \qquad \lim_{x \to +\infty} F(x;\lambda) = 1$$

  - ✓ The probability density function (pdf) f(x;λ): this function is the derivative of the cdf. In practice, this function gives the probability for the occurrence of a certain value. The common rule of all probability functions is applied for f(x;λ).
    - where λ > 0 is a parameter of the distribution, often called the rate parameter.
    - The distribution is supported on the interval [0,∞).

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & , x \geq 0, \\ 0 & , x < 0. \end{cases} \qquad f(x) = \frac{dF(x;\lambda)}{dx} \qquad F(x;\lambda) = \int_{-\infty}^{x} f(\tau)d\tau \qquad \int_{-\infty}^{+\infty} f(x)dx = 1$$

3.4. Some mathematical probability distribution functions (2)

- **Exercise:**
  1. Use the analytical expression of the CDF to derive the analytical expression of the PDF.
  2. Use the analytical expression of the PDF to derive the analytical expression of the CDF.
  3. Use the analytical expression of the PDF to calculate the mean/average
  4. Use the analytical expression of the PDF to calculate the variance
  5. Plot the PDF and CDF as functions of the time headway
  6. Comment the results obtained.
  7. Discuss the drawbacks of the exponential distribution

- **Solution:**
  1. Expression of the PDF using the CDF

$$f(x) = \frac{dF(x;\lambda)}{dx} \quad \Rightarrow \quad f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & , x \geq 0, \\ 0 & , x < 0. \end{cases}$$

3.4. Some mathematical probability distribution functions (3)

- **Solution:**

2. Expression of the CDF using the PDF

$$F(x;\lambda)=\int_{-\infty}^{x} f(\tau)d\tau \quad \Rightarrow \quad F(x;\lambda)=\begin{cases}1-e^{-\lambda x} & , x\geq 0,\\ 0 & , x<0.\end{cases}$$

3. Calculation of the mean/average ($\mu=\mu_1 \cong E(\underline{x})$)

$$\mu=\mu_1=\int_{-\infty}^{+\infty} xf(x)dx=\int_{-\infty}^{+\infty} xdF(x)=\int_{0}^{+\infty}\lambda xe^{-\lambda x}dx$$

$$E(\underline{x})=\lambda\int_{0}^{+\infty} x\frac{d\left[\frac{e^{-\lambda x}}{-\lambda}\right]}{dx}=\lambda\left[x\frac{e^{-\lambda x}}{-\lambda}\right]_0^{+\infty}-\lambda\int_{0}^{+\infty}\left[\frac{e^{-\lambda x}}{-\lambda}\right]dx=\int_{0}^{+\infty}e^{-\lambda x}dx=\left[\frac{e^{-\lambda x}}{-\lambda}\right]_0^{+\infty}=\frac{1}{\lambda}$$

3.4. Some mathematical probability distribution functions (4)

- **Solution:**

    4. Calculation of the variance

$$(V_2(\underline{x}) = \mathrm{var}(\underline{x}) = \sigma^2(\underline{x}) = \sigma^2 = \mu_2 - \mu^2)$$

$$\mu_2 = \int_{-\infty}^{+\infty} x^2 f(x)dx = \int_{-\infty}^{+\infty} x^2 dF(x) = \int_{0}^{+\infty} \lambda x^2 e^{-\lambda x} dx$$

$$= \lambda \int_0^{+\infty} x^2 \frac{d\left[\frac{e^{-\lambda x}}{-\lambda}\right]}{dx} = \lambda\left[x^2 \frac{e^{-\lambda x}}{-\lambda}\right]_0^{+\infty} - \lambda \int_0^{+\infty} 2x\left[\frac{e^{-\lambda x}}{-\lambda}\right]dx = 2\int_0^{+\infty} x e^{-\lambda x} dx = 2\left[\frac{\mu_1}{\lambda}\right] = \frac{2}{\lambda^2}$$

$$V_2(\underline{x}) = \mathrm{var}(\underline{x}) = \sigma^2(\underline{x}) = \sigma^2 = \frac{1}{\lambda^2}$$

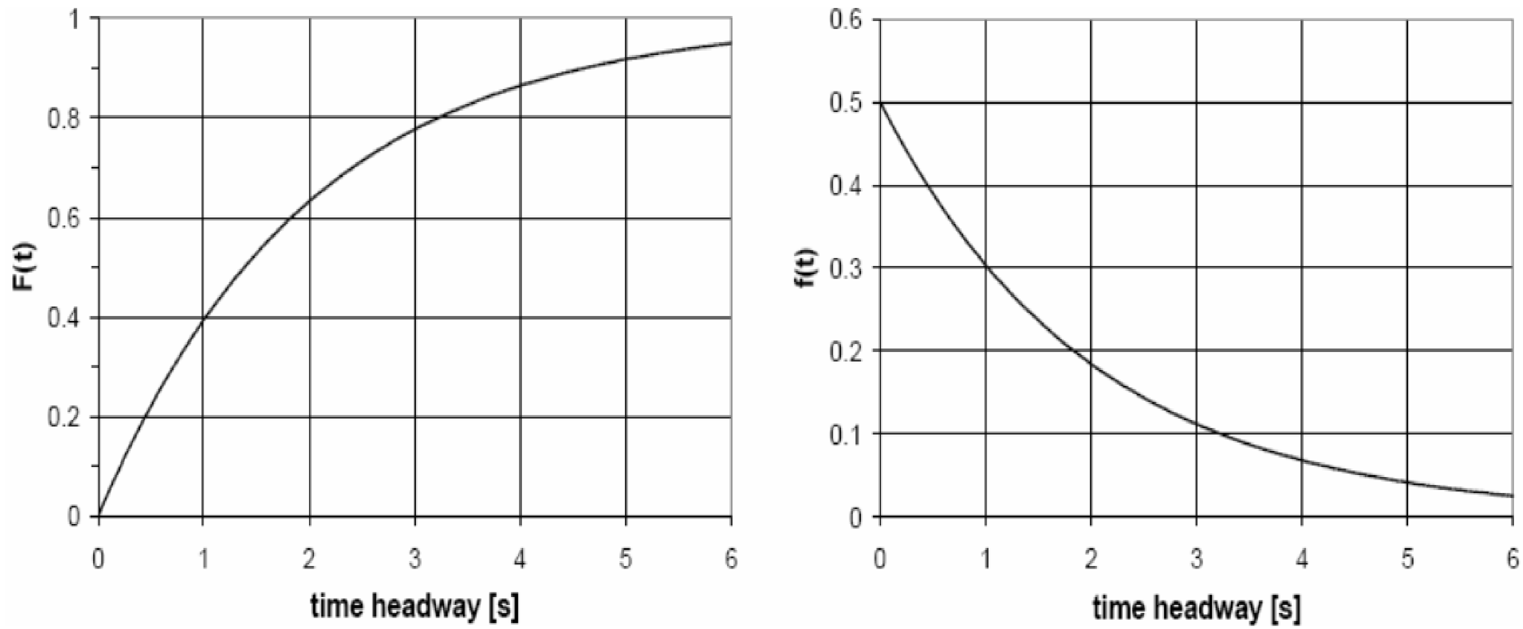
Various notations of the variance

- $V_2(\underline{x})$
- $\mathrm{var}(\underline{x})$
- $\sigma^2(\underline{x})$
- σ^2

3.4. Some mathematical probability distribution functions (5)

- **Solution:**

5. Plot of the PDF and CDF as functions of the time headway



3.4. Some mathematical probability distribution functions (6)

- **Solution:**

6. Comment of the plot of PDF and CDF as functions of the time headway

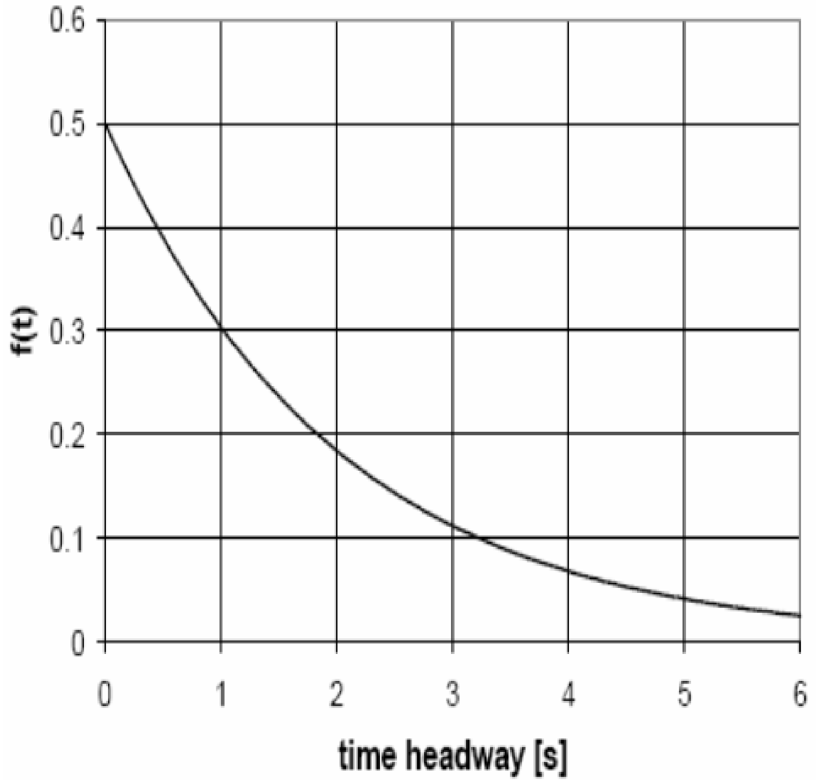
- ✓ The exponential distribution is a "memory-less" Markov process
  - E.g.: Having the condition that no car/train has passed during the time t0, the probability to wait for t more seconds before the next train/car passes is the same as the (unconditional) probability of a time headway t seconds
- ✓ The occurrence of an event is independent of the history
- ✓ The cdf is an increasing function of the time headway (gap time between packets/phone calls sent to a server or gap time between vehicles/trains crossing at a point)
- ✓ The cdf is asymptotically bounded
- ✓ The pdf is a decreasing function of the time headway
- ✓ The pdf is asymptotically bounded

3.4. Some mathematical probability distribution functions (7)

- **Solution:**

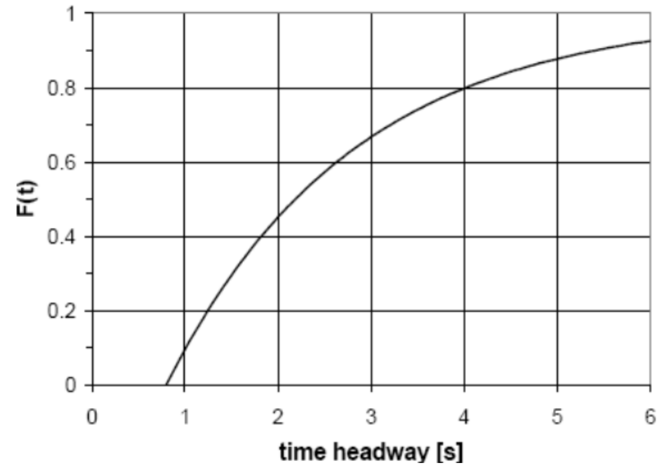
7. Drawbacks of the exponential distribution

- ✓ Small time headways have a relatively large probability.
- ✓ Time headway 0 is possible and has even the largest probability (Not realistic!)
- ✓ The distribution is defined with only one parameter $1/\lambda$
- ✓ When the average value is chosen, the variance is fixed and equal to $1/\lambda^2$



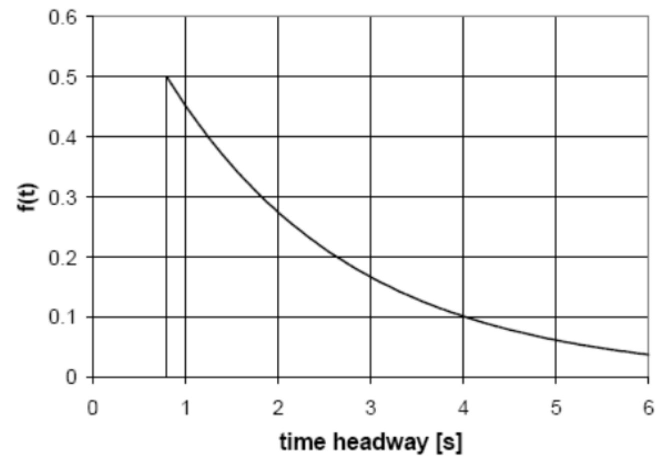
3.4. Some mathematical probability distribution functions (8)

- **Shifted exponential distribution**
  - ✓ This distribution overcomes the drawback of the exponential distribution.
  - ✓ The figures on this slides show the PDF and CDF as functions of the time headway.



$$F(x;\lambda) = \begin{cases} 1 - e^{-\lambda(x - x_0)} & , \ x \geq x_0 , \\ 0 & , \ x < x_0 . \end{cases}$$

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda(x - x_0)} & , \ x \geq x_0 , \\ 0 & , \ x < x_0 . \end{cases}$$



$$E(\underline{x}) = \frac{1}{\lambda} + x_0 \qquad \sigma^2 = \frac{1}{\lambda^2}$$

3.4. Some mathematical probability distribution functions (9)

- **Poisson distribution**
  - ✓ A **Poisson process** is a sequence of events "randomly spaced in time".
  - ✓ The Poisson distribution gives the probability P(k, t) that k flows (e. g. messages or vehicles, trains, …) are passing
    (or are sent) in a time t.
  - ✓ The probability that there are exactly k occurrences (k being a non-negative integer, k = 0, 1, 2, ...) is

$$f(k; \lambda t) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

  - ✓ e = 2.71828..., λ is a positive real number, equal to the expected number of occurrences that occur during the
    given interval.
    - E.g.: Suppose the events occurring on average every 4. If you are interested in the number of events
      occurring in a 10 minutes interval, you would use as model a Poisson distribution with λ = 10/4 = 2.5.

3.4. Some mathematical probability distribution functions (10)

- **Poisson distribution** (ctn.)
  - ✓ The probability that no measurements are lost during the registration time is

$$P(0,\tau)=e^{-\mu}$$

  - ✓ μ=λ.t is the average in time t.
  - ✓ μ is equal to the variance
  - ✓ The probability that no more than p measurements are lost is given by the following cumulative distribution:

$$P(0\leq x\leq p,\tau)=\sum_{k=0}^{p}\frac{(\lambda\tau)^k}{k!}e^{-\lambda\tau}=e^{-\lambda\tau}\sum_{k=0}^{p}\frac{(\lambda\tau)^k}{k!}=e^{-\mu}\sum_{k=0}^{p}\frac{(\mu)^k}{k!}$$

  - ✓ The average and variance are expressed as follows:

$$E(\underline{x})=\mu=\int_{-\infty}^{+\infty}xf(x)dx=\lambda \qquad \sigma^2=\lambda$$

3.4. Some mathematical probability distribution functions (11)

- **Poisson distribution** (ctn.)
  - Examples of Poisson model
    - Packets arriving to a buffer
    - Cars arriving in front of a traffic control panel
    - Customers arriving to a bank
- **Relation between Poisson and exponential distribution**
  - These two distributions are favorites for a first approximation when dealing with traffic flow issues.
  - An important property of the traffic stream is the density of flows (e.g. number of vehicles or packets of messages) that pass a cross section on a road (or server) in each time period/interval T
  - When in the interval T the average number of passing vehicles/packets is λ, then the probability that k vehicles/packets will pass in the interval T is given by the Poisson distribution

$$\Pr\{\underline{n} = k \mid T\} = \frac{(\lambda T)^k}{k!} e^{-\lambda T} \quad (k = 0, 1, 2 \ldots)$$

3.4. Some mathematical probability distribution functions (12)

- **Relation between Poisson and exponential distribution** (ctn.)
  - ✓ The probability that the time headway is larger than t, is the probability that in t seconds no vehicle/packet will pass. This probability is deduced for k=0, and is expressed as:

$$\Pr\{\underline{h} > t\} = \Pr\{\underline{n} = 0 \mid t\} = e^{-\lambda t}$$

  - ✓ From the precedings can be deduced the probability that the time headway is less than t, which is the probability that in t seconds vehicles/packets will pass.

$$\Pr\{\underline{h} \leq t\} = 1 - e^{-\lambda t}$$

  - ✓ The preceding demonstration has shown how the exponential distribution function can be obtained from the Poisson distribution function. This demonstration shows a straightforward relation between the two distribution functions.

3.4. Some mathematical probability distribution functions (13)

- **Poisson distribution of number of events per unit time vs Exponential distribution of waiting time until first occurrence of an event**
  - ✓ In the case where λ is taken to be the rate, i.e., the average number of occurrences per unit time, if *N.t* is the number of occurrences before time t then we have

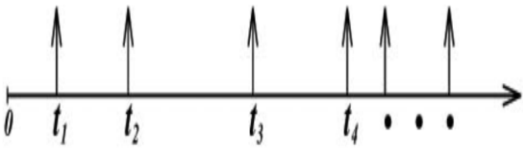
$$\Pr(N_t = k) = f(k; \lambda t) = \frac{e^{-\lambda t}(\lambda t)^k}{k!},$$

  - ✓ and the waiting time T until the first occurrence is a continuous random variable with an exponential distribution (with parameter λ). This may be deduced from the fact that

$$\Pr(T > t) = \Pr(N_t = 0) = e^{-\lambda t}.$$

  - ✓ Poisson process, Poisson distribution, exponential distribution
    - Number of events between time a and time b is given as $N_b - N_a$ and has a Poisson distribution
    - Inter-arrival time of event A and event B is given as $\tau_B - \tau_A$ and has an exponential distribution.

3.4. Some mathematical probability distribution functions (14)



- **Remark 1: Interarrival Times of a Poisson Process**
  - ✓ They are (pair-wise) independent
  - ✓ Each has an exponential distribution with mean $1/\lambda$
- **Remark 2: Markovian systems**
  - ✓ Distributions of A and B are exponential and thus exhibit the Markov (memory-less) property
  - ✓ Exponential distribution→ Markov property
  - ✓ Consequence of Markov property:
    - ○ State of the system can be summarized in a single variable, i.e. number of vehicles/packets/customers in the system
    - ○ Markovian systems are directly mappable to continuous-time Markov chain which can then be solved.
- **Remark 3: Markovian systems**
  - ✓ When the counting takes place on a single lane (e.g. A specific case of road traffic) the assumption that the measurements are Poisson distributed is not valid.
  - ✓ The appropriated distributions for such a traffic scenario are:
    - ○ The shifted negative exponential distribution
    - ○ The Erlang distribution

3.4. Some mathematical probability distribution functions (15)

- **Erlang distribution**
  - ✓ The exponential distribution offers only limited possibilities to approximate real time headways (i.e. times separating packets on the same line, vehicles/buses/trains on the same line, etc…)
  - ✓ The solution to overcome this limitation is:
    - The use of a distribution with more parameters (e.g. the Erlang-k distribution)
    - The use of a combination of distributions
  - ✓ The Erlang distribution is a Gamma distribution defined by the following cumulative probability function

$$E_k(t) = M^{k*}(t) = \int_0^t \frac{\left(\tau/\alpha\right)^{k-1}}{(k-1)!} e^{-(\tau/\alpha)} \, d\left(\tau/\alpha\right) = 1 - e^{-(t/\alpha)} \sum_{n=0}^{k-1} \frac{\left(t/\alpha\right)^n}{n!} \quad , t \geq 0$$

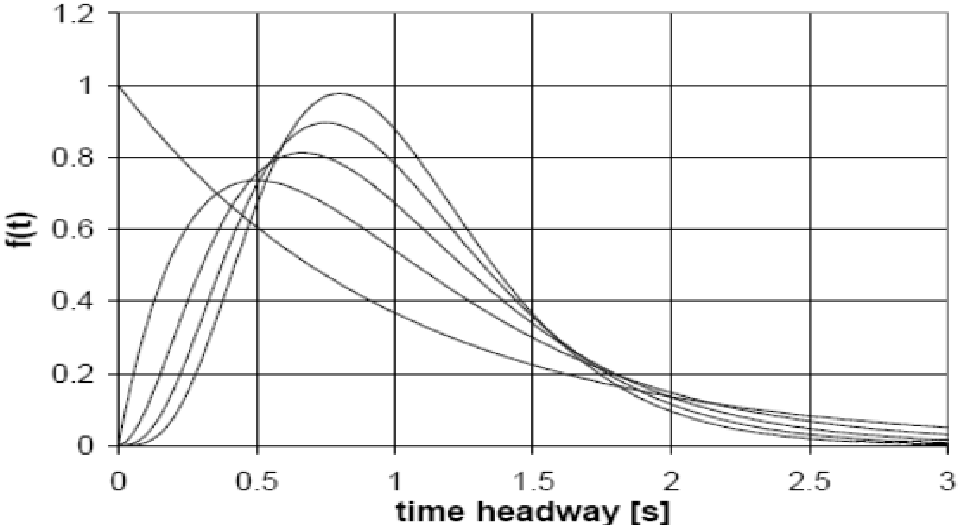
$$E_k(t) = 0 \quad , t < 0$$

3.4. Some mathematical probability distribution functions (16)

- **Erlang distribution: Average/mean and variance**
  - ✓ Using the analytical expression of the Erlang distribution function (see preceding slide), the average and variance can be calculated leading to following results:

$$f_k(t) = \frac{1}{\alpha} \frac{e^{-(t/\alpha)} \left(t/\alpha\right)^{k-1}}{(k-1)!}; \quad \mu = k \cdot \alpha; \quad \sigma^2 = k \cdot \alpha^2$$

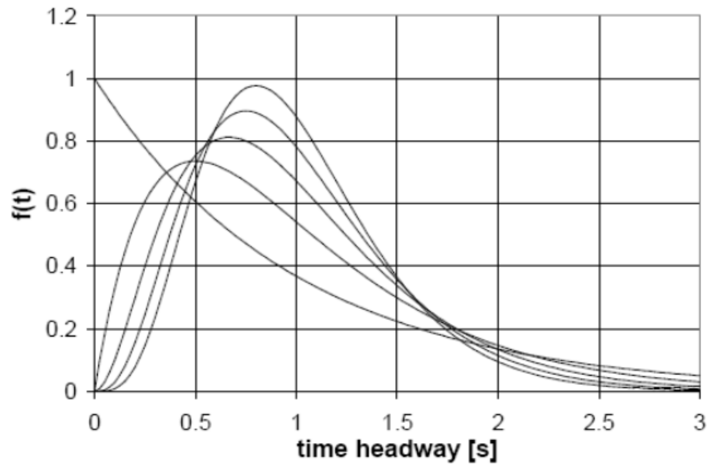
  - ✓ The figure below shows the Erlang distribution in terms of the time headway.



3.4. Some mathematical probability distribution functions (17)

- **Remarks**
  - ✓ The pdf of Erlang distribution is plotted in terms of the parameter k
  - ✓ For k=1, the Erlang distribution is exponential and has a pdf which tends to the unity.
  - ✓ When the parameter k increases (with constant average value $\mu = k\alpha$), the variance ($\sigma^2 = k\alpha^2$) will decrease, and the pdf tends to be symmetrical

$$f_k(t) = \frac{1}{\alpha} \frac{e^{-(t/\alpha)} (t/\alpha)^{k-1}}{(k-1)!}; \quad \mu = k \cdot \alpha; \quad \sigma^2 = k \cdot \alpha^2$$



3.5. Registration equipment (1)

- The registration of traffic data is a random/stochastic process
- The registration process can be conducted in two ways
  - ✓ The direct processing is possible if
    - The registration process is fast
    - The time between incoming measurements is much larger than the processing time (e.g. Counting the number of packets sent, or counting the number of passing vehicles, trains, ...)
  - ✓ The indirect processing (i.e. store first and process later).
- Traditionally, the probability of measurement loss depends upon the arrival distribution and the registration time as well.
- The current approach exploits electronic buffer memories to minimize the probability of data loss. The loss probability can still be calculated when using the mathematical description of the processes.

3.5. Registration equipment (2)

- There are various ways to organize a buffer memory in order to separate the input process from the registration process:
  - ✓ **1- A FIFO Buffer (First In, First Out)**
    - Measurements are stored in the memory in the order of arrival.
    - The registration process files/load locations in the memory progressively.
    - Buffer overflow is possible when the memory is full.
  - ✓ **2- A ring buffer**
    - This buffer consists of two pointers: the input pointer points the first empty location. The output pointer points to the measurement to be registered (the oldest measurement in memory). A new measurement is stored in the location pointed to by the input pointer; this pointer is shifted to the next empty location after storage.
    - The registration device processes the measurement that is pointed to by the output pointer.
    - The buffer is empty when input & output pointers indicate the same location.
    - Buffer overflow is possible when the input pointer passes the output pointer.
    - The advantage of this method is that after registration only the pointer must be moved, instead of all the measurements in the memory.

3.5. Registration equipment (3)

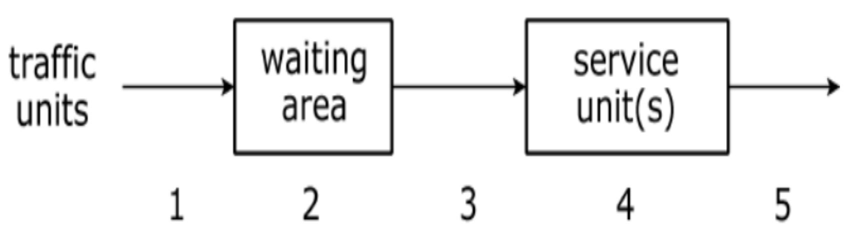
- ✓ **3- A double buffer**
  - The memory is divided into two equal parts: an input buffer that stores incoming measurements. A registration buffer that holds the measurements to be registered.
  - When the input buffer is full and the registration buffer is empty, the two parts change places.
  - The swapping time is very short to avoid loss of the measurements data.
  - Buffer overflow is possible when the input buffer is full before the registration buffer is completely processed.
- ✓ **4- A LIFO buffer (Last In, First Out)**
- ✓ **5- Random buffer**
- ✓ **6- Round robin buffer**
- ✓ **7- Priority buffer**
- ✓ **etc.....**
- ✓ **8- Important properties of a registration device**
  - **The burst rate**: i.e. the speed of the measurement storage (speed of the memory)
  - **The average rate**: i.e. the speed of the registration process

3.5. Registration equipment (4)

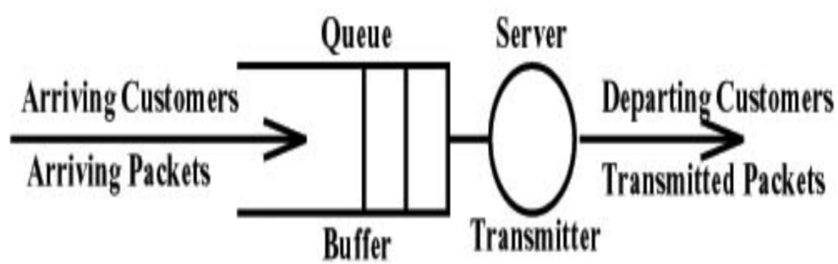
- ✓ **9- Two main groups for traffic measurements**
  - **Counting.** These measurements ensure storage of the number of events in each time
  - **Time measurements**. These measurements insure of the point in time (or a time interval) for the event.
- ✓ **10- Loss when counting.**
  - Registration devices without memories are exposed to the loss of measurements during the registration process
  - The Poisson distribution can be used to calculate the loss probability (see above)
  - For buffered devices, the loss probability is very small.
- ✓ **11- Loss when measuring interval times**
  - This can be caused by the buffer overflow
  - Queuing theory can be used to measure time intervals. This can be achieved by using the Poisson distribution and a fixed service time (registration time).
  - There is only one service station (registration device) to process the measurements (e.g. M\D\1 process see the part devoted to the queuing theory).
  - Characteristics of the M\D\1 system: (1) a Poisson arrival distribution (**M**), (2) a fixed service time (**D**) and (3) one service station (server) (**1**).

3.6. Overview of queuing and applications in transportation (1)

- **What is a queuing system?**
  - ✓ A traffic process, where traffic units (e.g. trains, vehicles, people, telephone calls,) request a certain service (e.g. connection, change of rail- trajectory, road- deviation, telephone connection, parking place, counter service, etc.) during a certain time.
  - ✓ A buffering of the traffic units in a waiting area is possible when the service elements (servers) are all occupied.
  - ✓ This task is performed specifically when the service is not available after arrival of the traffic units.
- **General and specific representations of a queuing system**



General representation



Specific representation

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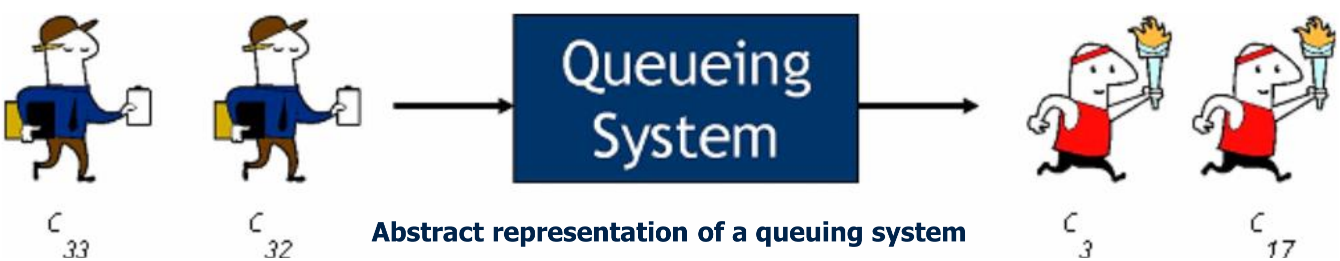
3.6. Overview of queuing and applications in transportation (2)

- **What are the main phases of a queuing process?**
  - ✓ **1. Arrival process:**
    - This phase is characterized in time by the distribution of the arrival intervals (time headway in railway/road traffic)
  - ✓ **2. Waiting area:**
    - This phase is characterized by the dimensions, like:
      - Infinite size;
      - Finite size: with the probability that traffic units are lost when the area is full;
      - Zero size: with the evidence that traffic units are lost; this is a loss system (e.g. systems without memory or without waiting area).
  - ✓ **3. Assignment strategy:**
    - This phase defines the policy i.e. the way in which available service stations are assigned to be waiting traffic units
      - **FIFO** “**F**irst **I**n, **F**irst **O**ut” (**FCFS** “**F**irst **C**ome, **F**irst **S**erve”)
      - **LIFO** “**L**ast **I**n, **F**irst **O**ut” (**LCFS** “**L**ast **C**ome, **F**irst **S**erve”)
      - **Random** (i.e, grading in automatic telephony systems)
      - **Assignment of different priorities** to different traffic categories

3.6. Overview of queuing and applications in transportation (3)

- ✓ **4. Service unit(s):**
  - This phase defines the place where the traffic units are served. The properties are:
    - Number of service units: this is an important property of the system (e.g. M/M/1,…..M/M/n, etc…)
    - Service duration: this is often a stochastic variable, characterized by the service duration distribution.
- ✓ **5. Output process:**
  - This phase is different from the arrival phase as it is influenced by the distributions of waiting time and service duration.

- ❑ **What are the elements of a queuing system?**
  - ✓ **1- Queue/Buffer**
  - ✓ **2- Server/Transmitter**



Abstract representation of a queuing system

3.6. Overview of queuing and applications in transportation (4)

- **Why queuing theory?**
  - ✓ Waiting in lines is an unpleasant task/phenomenon.
    - It can be time and/or energy consuming.
  - ✓ It is striking/interesting to understand the queuing process
    - Lines will get longer and longer in future
    - Understanding why things happen is useful
  - ✓ For computer networks: Queuing systems are a part of a broad class of dynamic systems
    - Also known as systems of flow
  - ✓ Flows in graphs are one example of dynamic systems
    - Flows change continuously with time
    - Flows can be steady or unsteady
    - Queuing theory analyses unsteady or random flow problems

3.6. Overview of queuing and applications in transportation (5)

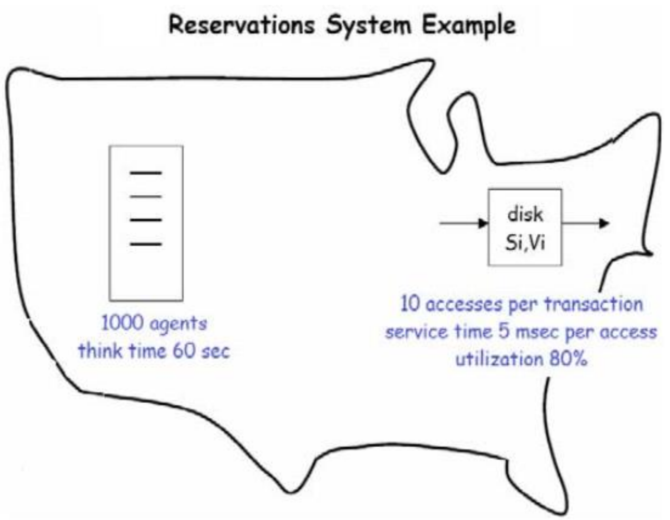
- **Some applications of queuing**
  - ✓ Telephone systems
  - ✓ Traffic flow
  - ✓ Communication systems
  - ✓ Communication networks
  - ✓ Trains/Vehicles waiting to be loaded or unloaded
  - ✓ Airplanes waiting for a runway
  - ✓ Jobs in a job shop
  - ✓ computer systems
  - ✓ machine plants, etc
- **Queuing theory answers questions like ..**
  - ✓ Mean waiting time in the queue
  - ✓ Mean system response time (waiting time in queue + service time)
  - ✓ Mean utilization of the servers
  - ✓ Distribution of the number of flows (e.g. telephone calls, Trains, vehicles, customers, etc.) in queue
  - ✓ Distribution of the number of flows (e.g. telephone calls, trains, vehicles, customers, etc.) in system
  - ✓ Probability of the queue is full or empty

3.6. Overview of queuing and applications in transportation (6)

- **First illustrative example where queuing theory is being used (1)**
  - **Consider a computing center for managing jobs**
    - Jobs are submitted to be executed "later".
    - Arrivals for new requests are unpredictable
    - Execution times are unpredictable
  - **Queuing theory provides answers to following questions**
    - How long may a job expect to wait before getting service?
    - How many jobs will be served before the one just submitted?
    - For what fraction of the day will the computing center be busy?
    - How long will the intervals of continuous busy work be?
- **Second illustrative example where queuing theory is being used (2)**
  - **Exercise. "***Train/Bus/Airline reservation example"*
    - 1000 reservation agents around Austria
    - "Disk farm" in Klagenfurt city
    - Each agent issues new transactions against the database every 60 seconds
    - Every transaction accesses the directory disk an average of 10 times

3.6. Overview of queuing and applications in transportation (7)

- ✓ **Exercise.** “*Train/Bus/Airline reservation example*” (ctn.)
  - The directory disk takes an average of 5 milliseconds to serve a request and is in use 80% of the time
  - What is the throughput (jobs per second) completed by the entire system?
  - What is the response time experienced by an agent waiting for a transaction?
  - Can these questions be answered precisely?, approximately? or not at all?
- ✓ Queuing theory gives the basic tools for answering such questions
- ✓ The theory deals with randomness in physical processes such
  - The arrival times of agent requests
  - The service times at the disks and CPUs
  - Lengths of queues
  - Variations in response times
- ✓ The theory allows us to characterize the performance measures statistically, in terms of averages, given the statistics of arrivals and services.



3.6. Overview of queuing and applications in transportation (8)

- **Fundamentals of Queuing Systems**
  - ✓ **Five main points of interest**
    - **Arrival process: How Packets/Trains/Vehicles/Customers arrive?**
      - A(n,t)= arrivals that see n customers in a system during [0, t]
      - A(t)= total arrivals during [0, t]
      - Typically described as probability distribution of inter arrival times
    - **Service process: How much demand do requests generate?**
    - **How many service stations?**
    - **How many places in queue?**
    - **How are queues processed?**
  - ✓ **Key metrics to be determined**
    - Waiting time for a **Phone call/ Packet/Vehicle/Train/Customer**
    - Number of **Phone calls/ Packets/Vehicles/Customers** in the system
    - Length of busy and idle periods
    - Amount of work backlog (undone work)

3.7. General queuing notation (Ref. Kendall 1951) (1)

- **Five components of a Queuing system**
  - ✓ 1. Interarrival-time probability density function (pdf)
  - ✓ 2. service-time pdf
  - ✓ 3. Number of servers
  - ✓ 4. queuing discipline
  - ✓ 5. size of queue.

- **These five components influence the general queuing notation (Kendall)**
  - ✓ **A/B/m/N –S**
    - **A** – distribution of inter-arrival time
    - **B** – distribution of service time
    - **m** – number of servers
    - **N** – max size of the waiting line (∞ if omitted)
    - **S** – queue discipline (FIFO if omitted)

3.7. General queuing notation (Ref. Kendall 1951) (2)

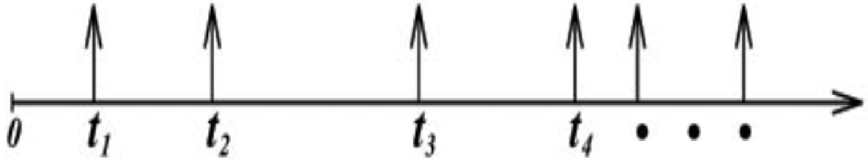
General (Ref. Kendall 1951) notation: X/Y/m/k

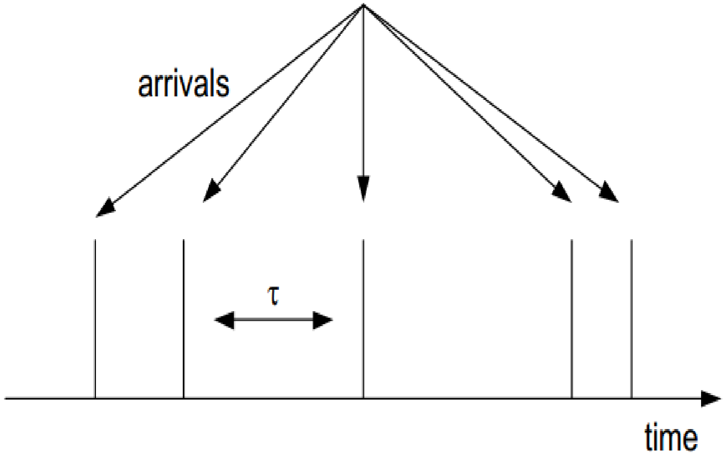
- **X** indicates the distribution of the arrival intervals (or **X** is a symbol representing the interarrival process)
  - ✓ "M" (Markov)=Poisson (exponential interarrival times τ, i.e. times separating successive arrivals)
  - ✓ "D" = Deterministic (constant τ).
- **Y** indicates the distribution of the service durations (or **Y** is a symbol representing the service distribution)
  - ✓ "M" (Markov) = exponential,
  - ✓ "D" = deterministic
  - ✓ "G" = General (or arbitrary)
- **m** = indicates the number of service units (i.e. the number of servers)
- **k** = number of buffer slots (omitted when k = ∞)

3.7. General queuing notation (Ref. Kendall 1951) (3)

Interarrival Times of a Poisson Process

- The interarrival time random variables
  - ✓ τ_1 is the time untill the first arrival
  - ✓ τ_2 is the time between the first and second arrivals
  - ✓ τ_3 is the time between the second and third arrivals
  - ✓ τ_1, τ_2 , and τ_3 are pairwise independent.
  - ✓ Each has an exponential distribution with mean 1/λ.





3.8. Some queuing systems/models

Choices for inter-arrival time (A) and service time (B)

- **A** and **B** can be:
  - ✓ M (Markov) – exponential distribution
  - ✓ D (Deterministic) – constant value
  - ✓ Ek (Erlang –k) – Erlangian distribution
  - ✓ Hk (Hyper-k) – hyper-exponential distribution
  - ✓ General – general distribution

- Various types of queuing models/systems
  - ✓ M/M/1 queuing models/system
  - ✓ M/M/n queuing models/system
  - ✓ D/D/1 queuing model/system
  - ✓ M/D/1 queuing model/system
  - ✓ G/G/1 queuing model/system and Little's law
  - ✓ Etc....

3.8. Some queuing systems/models

The M/M/1 Queue

- **M/M/1 queuing model/system**
  - ✓ Characteristics of the model
    - Model/System made-up of a single queue
    - Model/System made-up of a single server
    - Model/System having a single traffic source
    - Model/System having an infinite storage capacity
    - FIFO policy is used for queue management
  - ✓ Notation of the model/system
    - ✓ M stands for Markovian
    - ✓ M/M/1 means that the system has:
      - A Poisson arrival process
      - An exponential service time distribution
      - A single server
      - infinite storage capacity
      - FIFO policy

3.8. Some queuing systems/models

The D/D/1 Queue

- **The D/D/1 queue has**
  - ✓ Deterministic (constant/fixed) arrivals (periodic with period = $1/\lambda$)
  - ✓ Deterministic service times (each service takes exactly $1/\mu$)
  - ✓ 1 server
  - ✓ infinite storage capacity (Infinite length buffer)
  - ✓ FIFO policy

- If $\lambda < \mu$ then there is no waiting in a D/D/1 queue

- Randomness is a major cause of delay

3.8. Some queuing systems/models

The M/D/1 Queue

- **The M/D/1 queue has**
  - ✓ Poisson arrivals process (with rate λ)
  - ✓ Deterministic service times (i.e. server with constant/fixed service time)
  - ✓ 1 server
  - ✓ Infinite length buffer
  - ✓ FIFO policy

The G/G/1 Queue

- **The G/G/1 queue has**
  - ✓ Poisson arrivals process governed by a general (or arbitrary) distribution
  - ✓ Service times governed by a general (or arbitrary) distribution
  - ✓ 1 server
  - ✓ Infinite length buffer
  - ✓ FIFO policy

3.9. State Analysis of Queue models/systems

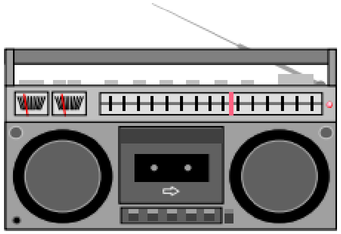
Poisson's law

$$P_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t} \qquad (1)$$

3.9. State Analysis of Queue models/systems

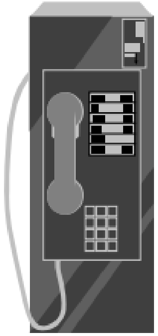
Poisson's law in physics

- radio active decay
  - P[k alpha particles in t seconds]
  - λ = average nbr of prtcls per second



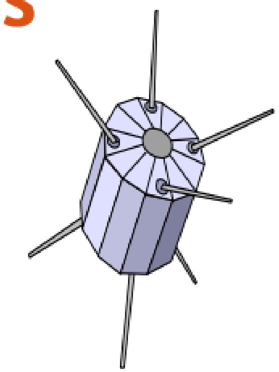
Poisson's law in operations research

- planning switchboard sizes
  - P[k calls in t seconds]
  - λ =avg number of calls per sec



3.9. State Analysis of Queue models/systems

Poisson's law in communications



- in designing a fiber optic xmit-rcvr link
  - P[k photoelectrons generated at the rcvr in one second]
  - λ=avg number of photoelectrons per sec.

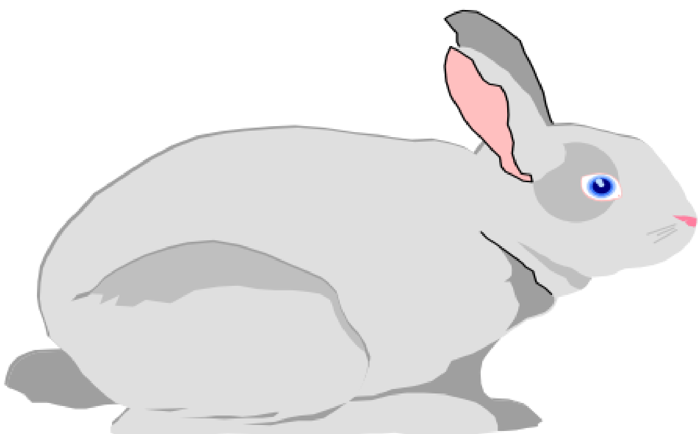
Poisson's law in transportation

- planning size of highway tolls
  - P[k autos in t minutes]
  - λ =avg number of autos per minute

3.9. State Analysis of Queue models/systems

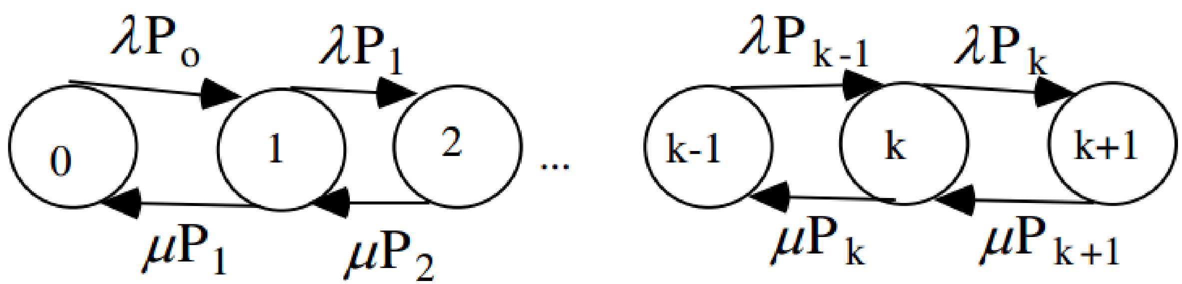
λ - Rate parameter

λ =event per unit interval (time distance volume...)



3.9. State Analysis of Queue models/systems

- **Single-server queuing system**



λ = mean arrival rate (cust./sec)

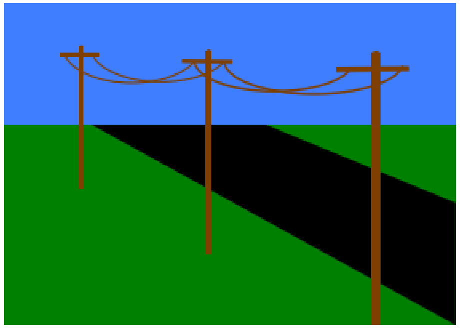
λP_0 = mean number of transitions/ sec from state 0 to 1

μ = mean service rate (cust./sec)

μP_1 = mean number of transitions/ sec from state 1 to 0

3.9. State Analysis of Queue models/systems

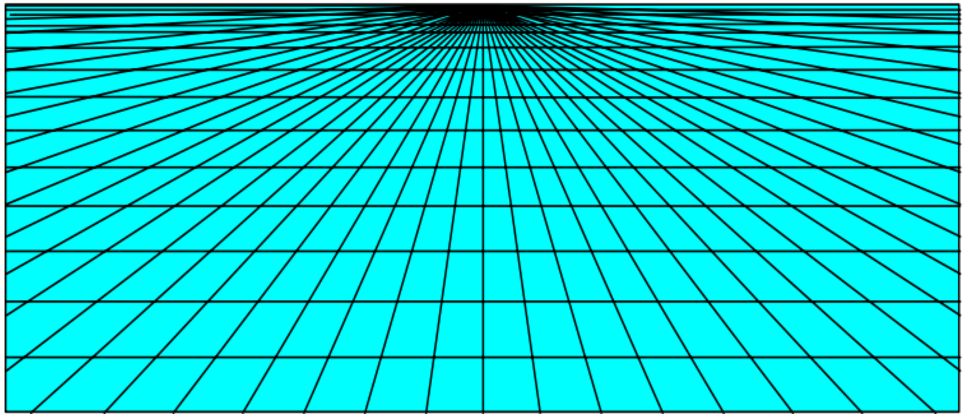
- **State**
  - ✓ State 0 = system empty
  - ✓ State 1 = Pack./cust. in server
  - ✓ State 2 = Pack./cust. in server, 1 cust. in queue etc...



- **Probability of given state**
  - ✓ Prob. of a given state is invariant wrt time if system is in equilibrium.
  - ✓ The prob. of k Pack./cust's in system is constant.

3.9. State Analysis of Queue models/systems

- **Similar to AC**
  - ✓ This is like AC current entering a node
  - ✓ is called detailed balancing
  - ✓ the number leaving a node must equal the number entering



3.9. State Analysis of Queue models/systems

- **Derivation**

(2) $\lambda P_0 = \mu P_1$

(2a) $P_1 = \dfrac{\lambda P_0}{\mu}$

(3) $\lambda P_1 = \mu P_2$

(3a) $P_2 = \dfrac{\lambda P_1}{\mu}$

$$\lambda P_k = \mu P_{k+1}$$

$$P_k = \frac{\lambda^k P_0}{\mu^k} = \rho^k P_0$$

where $\rho = \dfrac{\lambda}{\mu}$ **= traffic intensity < 1**

3.9. State Analysis of Queue models/systems

- **All probability sum equal 1:**

$$P_k = \frac{\lambda^k P_0}{\mu^k} = \rho^k P_0$$

$$\sum_{k=0}^{\infty} \rho^k P_0 = 1 = P_0 \sum_{k=0}^{\infty} \rho^k = 1$$

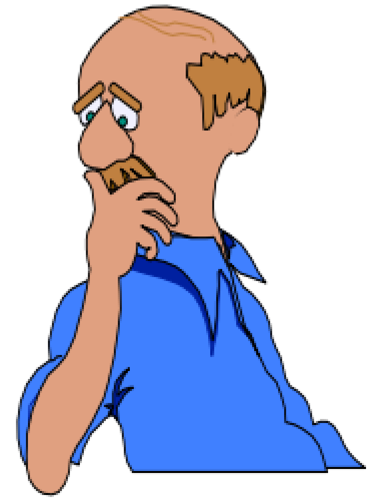
- **geometric sum:**

$$\sum_{k=0}^{\infty} \rho^k = \frac{1}{1-\rho} \qquad \left(\rho = \frac{\lambda}{\mu} < 1\right)$$

- **Finally:**

$$P_0 = 1 - \rho$$ **= prob. server is empty**

$$P_k = (1-\rho)\rho^k$$



3.9. State Analysis of Queue models/systems

- **Mean value:**
  - ✓ let N=mean number of pack's/cust's in the system
  - ✓ To compute the average (mean) value use:

$$E[k]=\sum_{k=0}^{\infty} kP_k$$

- **Differentiation wrt k:**

$$D_k\sum_{k=0}^{\infty}\rho^k = D_k\frac{1}{1-\rho}=\sum_{k=0}^{\infty}k\rho^{k-1}=\frac{1}{(1-\rho)^2}$$

- **Finally:**

$$E[k]=N=(1-\rho)\frac{\rho}{(1-\rho)^2}=\frac{\rho}{(1-\rho)}$$

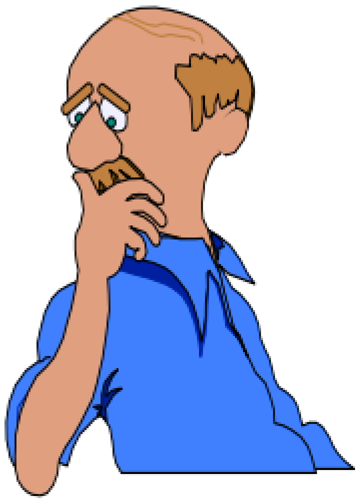
3.9. State Analysis of Queue models/systems

- **Application: Case of M/M/1 queue (1):**
  - Arrival Rate- λ (customer / min)
  - Server Utilization $\rho = \lambda / \mu$
  - Probability of 2 or more customers in system

    $P[N \geq 2] = \rho^2$
  - Mean steady-state number in the system

    $L = E[N] = \rho / (1 - \rho)$
  - S.D. of number of customers in the system

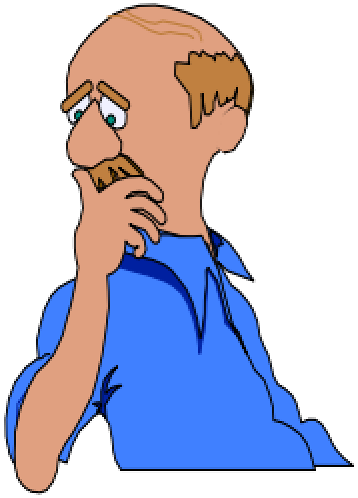
    $\sigma_N = sqrt(\rho) / (1 - \rho)$



3.9. State Analysis of Queue models/systems

- **Application: Case of M/M/1 queue (2):**

  - Mean time a customer spends in the system
    $W = E[w] = E[s] / (1 - \rho)$
  - S.D. of time a customer spends in the system
    $\sigma_w = E[w]$
  - Mean steady-state number of customers in queue
    $L_q = \rho^2 / (1 - \rho)$
  - Mean steady-state queue length of nonempty Qs
    $E[N_q \mid N_q > 0] = 1 / (1 - \rho)$
  - Mean time in queue
    $W_q = E[q] = \rho \times E[s] / (1 - \rho)$



3.9. State Analysis of Queue models/systems

- **Proposal of some exercises based on concrete application examples to be solved by students**

Exercises must be Proposed by the Lecturer

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3.10. Blocking queuing Models/systems

- **What is an Erlang?**
  - ✓ 1. A unit of telecommunications traffic measurement
    - Describes the **total traffic volume of one hour**
  - ✓ 2. A representation of the **continuous use** of one voice path
  - ✓ 3. A quantity of traffic expressed in a unit time (1 hour)
- **Example**
  - ✓ A group of user made 30 calls in one hour. Each call had an average call duration of 5 minutes. Evaluate the number of Erlangs.
- **Solution**
  - ✓ (2.) → (30x5minutes)=150 minutes (of continuous use)
  - ✓ (1.) → (150minutes)/(60minutes)=2.5 Erlangs (i.e. traffic volume of one hour)
- **Remark**
  - ✓ **(2.5 Erlangs) -→ 2.5 hours of continuous use**
  - ✓ During the 1940s, the Erlang became the accepted unit of telecommunication traffic measurement,
  - ✓ The Erlang is still used today (as a unit of traffic) in the design of modern telecommunications networks.

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3.10. Blocking queuing Models/systems

- **What is the goal/interest of Erlang traffic measurements?**
  - ✓ Help telecommunications network designers to understand traffic patters/tendencies within their voice network
    - Facilitates the design of the network topology
    - Facilitates the establishment of the necessary trunk group **(i.e. a group of parallel connections)** sizes
  - ✓ Can be used to evaluate the number of lines required (or necessary)
    - Between a telephone system and a central office or
    - Between multiple networks locations
  - ✓ Can be exploited basically during Busy Hour Traffic (BHT) to
    - Block specific telephones/telecommunications calls
    - Give priority to specific telephones/telecommunications calls

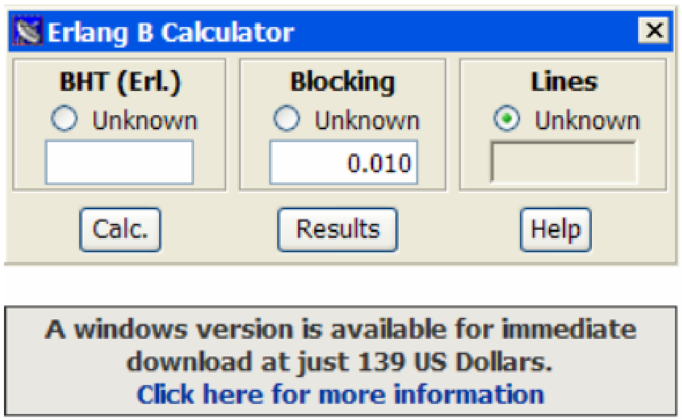
3.10. Blocking queuing Models/systems

- **Why the Erlang traffic models?**
  - ✓ **Design** and **optimization** of the traffic network
    - Estimation of the **number of lines**
    - Estimation of the **Busy Hour Traffic (BHT in Erlangs)**
    - Estimation of the **Blocking** which is the failure of calls due to an insufficient number of lines being available (e.g. 0.03 mean 3 calls blocked per 100 calls attempted)
    - Modelling of **queuing** situations/processes
    - Estimation of the **requirements/needs of call centers** (numb. of agents needed)

- **What are the types of Erlang traffic models?**
  - ✓ Three main types of Erlang traffic model
    - **Erlang B model**
    - **Extended Erlang B model**
    - **Erlang C model**

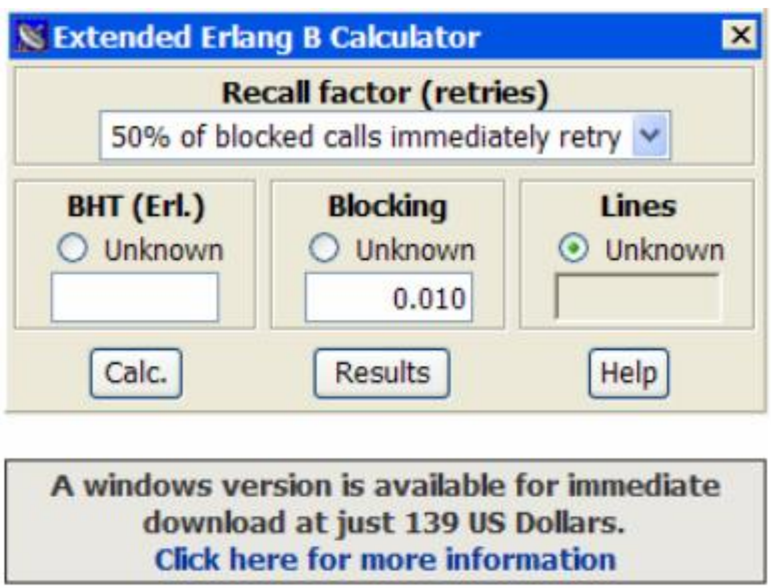
3.10. Blocking queuing Models/systems

- **What are the advantages of Erlang traffic models?**
  - ✓ Formulae exist to estimate the number of lines required in a telecommunications network
  - ✓ Formulae exist to estimate the number of lines required to a central office (e.g. Public Switched Telephone Network (PSTN) exchange lines)
  - ✓ Formulae exist to model queuing situations/processes/phenomena
  - ✓ Formulae exist to estimate the requirements or needs of call centers
- **Erlang B model:**
  - ✓ Is the most commonly used traffic model
  - ✓ Is used to determine the number of necessary lines (or lines needed) if the traffic estimation (in Erlangs) during the busiest hour is known
  - ✓ This model assumes that all blocked calls are immediately cleared



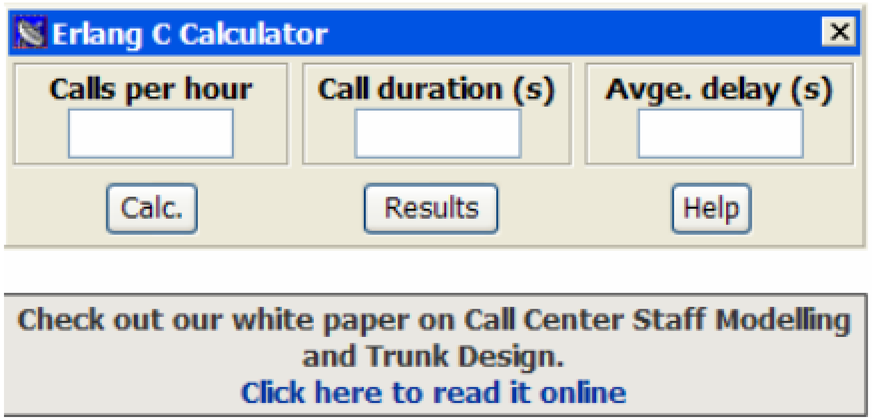
3.10. Blocking queuing Models/systems

- **Extended Erlang B model:**
  - ✓ This model **is very similar to Erlang B** with the following extension:
    - It considers that a percentage of calls are immediately represented to the system if they encounter blocking (a busy signal). This percentage can be specified



3.10. Blocking queuing Models/systems

- **Erlang C model:**
  - ✓ This model assumes that all blocked calls stay in the system until they can be processed/handled (queuing process).
  - ✓ This model can be applied to the design of call centers with the following functioning principle:
    - If calls cannot be answered immediately, they enter a queue (delay process)



3.10. Blocking queuing Models/systems

- **What are the Uses/Exploitations of the Erlang B traffic model?**
  - ✓ Exploited to estimate the number of lines required for the PSTN connections
  - ✓ Exploited to estimate the number of lines required for private wire connections
    - Three parameters/variables are involved:
      - **Busy Hour Traffic (BHT in Erlangs)** which is the number of hours of call traffic during the busiest hour of operation of a telephone system
      - **Blocking** which is the failure of calls due to an insufficient number of lines being available (e.g. 0.03 mean 3 calls blocked per 100 calls attempted)
      - **Lines** is the number of lines in a trunk group (i.e. a group of parallel connections)
  - ✓ **Application example:**
    - Suppose from your call logger you know that the BHT is 10 Erlangs, and you want to know how many lines are required in this trunk group if you are prepared to tolerate 2 calls being blocked in every 100 calls attempted.
      - The Erlang B calculator is appropriate to solve this problem.
      - The results from this calculator shows that 17 lines would be required for this trunk group.

3.10. Blocking queuing Models/systems

- **What are the Uses/Exploitations of the extended Erlang B traffic model?**
  - ✓ Exploited to estimate the number of lines required for the PSTN connections (CO trunks)
  - ✓ Exploited to estimate the number of lines required for private wire connections
  - ✓ This model takes into account the additional traffic load caused by blocked callers immediately trying to call again if their calls are blocked
    - Four parameters/variables are involved:
      - **Recall factor** is the percentage of calls which immediately retry if their calls are blocked
      - **Busy Hour Traffic (BHT in Erlangs)** which is the number of hours of call traffic there are during the busiest hour of operation of a telephone system
      - **Blocking** which is the failure of calls due to an insufficient number of lines being available (e.g. 0.03 mean 3 calls blocked per 100 calls attempted)
      - **Lines** is the number of lines in a trunk group (i.e. a group of parallel connections)

3.10. Blocking queuing Models/systems

- **Application example:**
  - ✓ Suppose that 40% of blocked calls are immediately retried. Also suppose that from your call logger the BHT is 10 Erlangs, and you want to know how many lines are required in this trunk group if you are prepared to tolerate 2 calls being blocked in every 100 calls attempted.
    - The extended Erlang B calculator is appropriate to solve this problem
    - The results from this calculator shows that 17 lines would be required for this trunk group
- **What are the Uses/Exploitations of the Erlang C traffic model?**
  - ✓ Exploited to **estimate the number of agents needed** in a given call center during an hour
    - The parameters required for this estimation are:
      - The number of calls received during that hour
      - The average duration of those calls
      - The average delay to be tolerated in answering all calls
      - Exploited to establish the number of switchboard operators required

3.10. Blocking queuing Models/systems

- **The Erlang B distribution**
  - ✓ The Erlang B distribution is used for dimensioning server pools where requests for service wait on a first in, first out (FIFO) queue until an idle server is available. It is based on the following assumptions:
    - There are an infinite number of sources;
    - Calls arrive at random;
    - Calls are served in order of arrival;
    - Blocked calls are delayed; and
    - Holding times are exponentially distributed.
  - ✓ The Erlang B distribution model is very robust to the traffic process
    - The distribution is very successful

3.10. Blocking queuing Models/systems

- **The Erlang B distribution**
  - ✓ The Erlang B formula is used to predict the probability that a call will be delayed. The Erlang C formula is:

$$E_N(A) = P(>0) = \frac{\frac{A^N}{N!}}{\sum_{i=0}^{N} \frac{A^i}{i!}}$$

  - ✓ where:
    - ○ A=Traffic offered to group in Erlangs (or User traffic described by offered traffic A)
    - ○ N=Number of servers in full availability group (or Network described by number of channels N)
    - ○ $E_N(A)=P(>0)$=Probability of delay greater than zero (or Quality-of-service described by blocking probability E)

3.10. Blocking queuing Models/systems

- **The Erlang C distribution**
  - ✓ The Erlang C distribution is used for dimensioning server pools where requests for service wait on a first in, first out (FIFO) queue until an idle server is available. It is based on the following assumptions:
    - There are an infinite number of sources;
    - Calls arrive at random;
    - Calls are served in order of arrival;
    - Blocked calls are delayed; and
    - Holding times are exponentially distributed.

3.10. Blocking queuing Models/systems

- **The Erlang C distribution**
  - ✓ The Erlang C formula is used to predict the probability that a call will be delayed and can be used to predict the probability that a call will be delayed more than a certain time. From that other key queue performance metrics can be calculated. The Erlang C formula is:

$$P(>0) = \frac{\dfrac{A^N N}{N!\,(N-A)}}{\displaystyle\sum_{i=0}^{N-1} \frac{A^i}{i!} + \frac{A^N N}{N!\,(N-A)}}$$

  - ✓ where:
    - P(>0)=Probability of delay greater than zero
    - N=Number of servers in full availability group
    - A=Traffic offered to group in Erlangs

3.10. Blocking queuing Models/systems

- **What is the Busy Hour Traffic (BHT)?**
  - ✓ The quantity of traffic expressed in a unit called Erlang
  - ✓ The number of hours of traffic offered to a trunk group
    - 1 Erlang can be considered equivalent to 1 hour of calls

  - ✓ Example:
    - A call logger indicates that 350 calls are made on a trunk group, and the average
      call duration 180 seconds. What is the Busy Hour Traffic (BHT)?.

  - ✓ Solution:
    - BHT=17.5 Erlangs

3.10. Blocking queuing Models/systems

- **What is Blocking describes?**
  - ✓ The number or percentage of calls which cannot be completed because insufficient lines have been provided.
    - 0.01 means 1% of calls will be blocked (this is the normal rate in traffic engineering)
    - 0.03 (i.e. 3%) blocking can be used for specific applications
- **What are the key factors to estimate the number of lines required?**
  - ✓ The BHT and
  - ✓ Blocking
    - The Erlang B calculator considers these two parameters for the estimation
    - Example
      - BHT=17.986Erlangs & Blocking=0.01 leads to the requirement of **28 lines**
      - BHT=17Erlangs & Blocking=0.01 leads to the requirement of **27 lines**
      - BHT=17Erlangs & Blocking=0.03 leads to the requirement of **24 lines**

Chapter 4.

Approximation and fitting

(Chapter’s detailed description)

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4.1. Norm approximation

- ✓ Introducing the basic norm approximation, the penalty function approximation and the approximation with constraints.

4.2. Least-norm problems

- ✓ Presentation of the Least-norm problems.

4.3. Regularized approximation

- ✓ Description of the concepts of Bi-criterion formulation, Regularization, Reconstruction, smoothing, and de-noising.

4.4. Robust approximation

- ✓ Analysis of Stochastic robust approximation, and Worst-case robust approximation.

4.5. Function fitting and interpolation

- ✓ Description of the following concepts: * Function families, * Constraints, * Fitting and interpolation problems, * Sparse descriptions and basis pursuit, * Interpolation with convex functions.

4.5. Application exercises

- ✓ Several exercises are proposed here by the lecturer to be solved by students

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Chapter 5.

Statistical estimation

(Chapter’s detailed description)

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5.1. Parametric distribution estimation

- ✓ Presentation of the "Maximum likelihood estimation" and "Maximum a posteriori probability estimation".

5.2. Nonparametric distribution estimation

- ✓ Presentation of the Nonparametric distribution estimation

5.3. Optimal detector design and hypothesis testing

- ✓ Presentation of deterministic and randomized detectors; Detection probability matrix; Optimal detector design; Multicriteria formulation and scalarization; Binary hypothesis testing; Robust detectors.

5.4. Chebyshev and Chernoff bounds

- ✓ Description of the concepts of Chebyshev bounds, and Chernoff bounds; presentation of some concrete application examples for illustration

5.5. Experiment design

- ✓ Presentation of the methodology of the experimental design

5.5. Application exercises

- ✓ Selected exercises with solutions are proposed in this section for illustration

Chapter 6.

Geometric problems

(Chapter’s detailed description)

6.1. Projection on a set

- ✓ Analysis of the projection of a point on a convex set; separating a point and a convex set; Projection and separation via indicator and support functions.

6.2. Distance between sets

- ✓ Analysis of distance between sets: Computing the distance between convex sets; Separating convex sets; Distance and separation via indicator and support functions

6.3. Euclidean distance and angle problems

- ✓ Study of selected concepts and problems: Gram matrix and realizability; Problems involving angles only; Euclidean distance problems.

6.4. Extremal volume ellipsoids

- ✓ Study of * the Löwner-John ellipsoid, * Maximum volume inscribed ellipsoid, and * Affine invariance of extremal volume ellipsoids.

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6.5. Centering

- ✓ Analysis of selected concepts for centering: * Chebyshev center; * Maximum volume ellipsoid center; * Analytic center of a set of inequalities

6.6. Classification

- ✓ Study of linear and nonlinear classification techniques.

6.7. Placement and location

- ✓ Study of linear and nonlinear location problems without constraints. Extension to the study of location problems with path constraints,

6.8. Floor planning

- ✓ Study of selected case studies: Relative positioning constraints; Floor planning via convex optimization; Floor planning via geometric programming

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Chapter 7.

Selected real-world scenarios as application examples in transportation

(Chapter's detailed description)

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7.1. Supply Chain Networks - Selected concrete scenarios are:

- ✓ Dynamic supply chains with stochastic policies
- ✓ Dynamic supply chains with stochastic demands
- ✓ Modelling of a supply chain network driven by stochastic fluctuations

7.2. Railway transportation

- ✓ Stochastic analysis of dynamic interaction between train and railway turnout
- ✓ Simulation of Train Track Interaction with Stochastic Track Properties
- ✓ Stochastic modeling of track irregularities using experimental measurements

7.3. Road transportation

- ✓ Application of Stochastic Modeling and Simulation to Vehicle System Dynamics
- ✓ Stochastic Modeling and Simulation of Traffic Flow
- ✓ Stochastic modelling of traffic flow and corresponding models
- ✓ Traffic flow theory and chaotic behavior

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END
OF
LECTURE 3

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